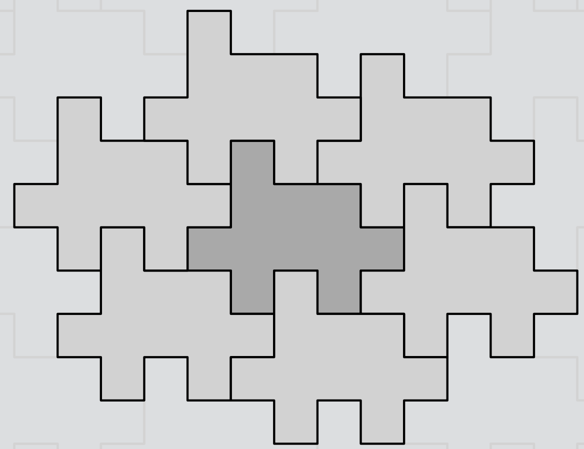
# An Optimal Algorithm for Tiling the Plane with a Translated Polyomino



Andrew Winslow

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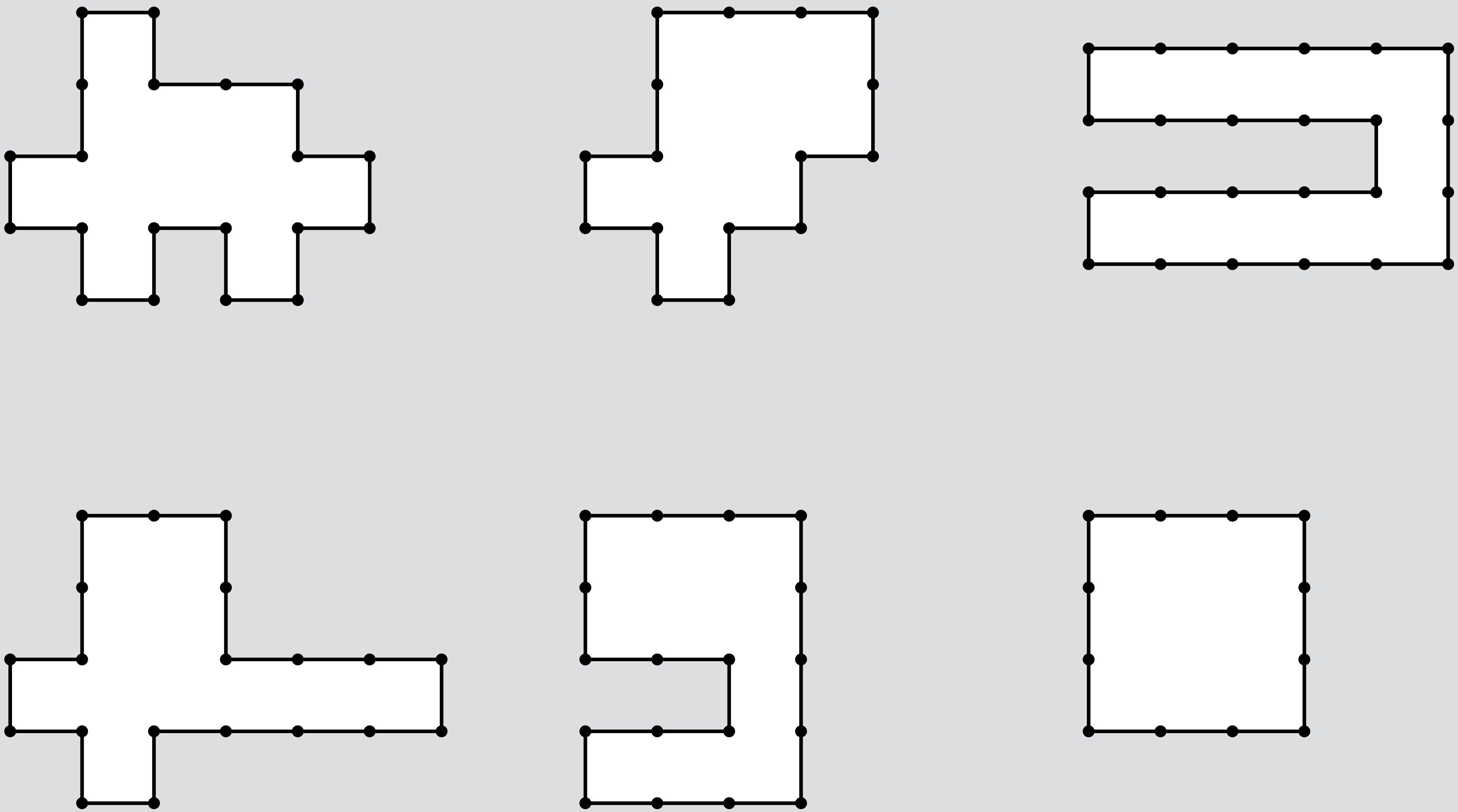
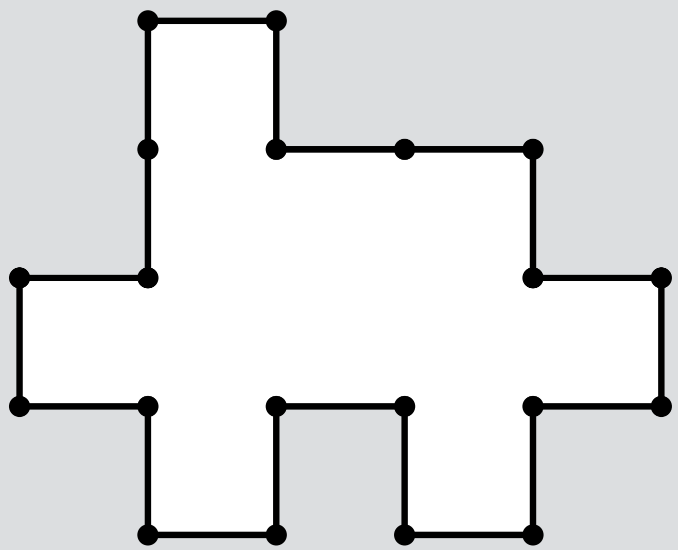
# Polyominoes

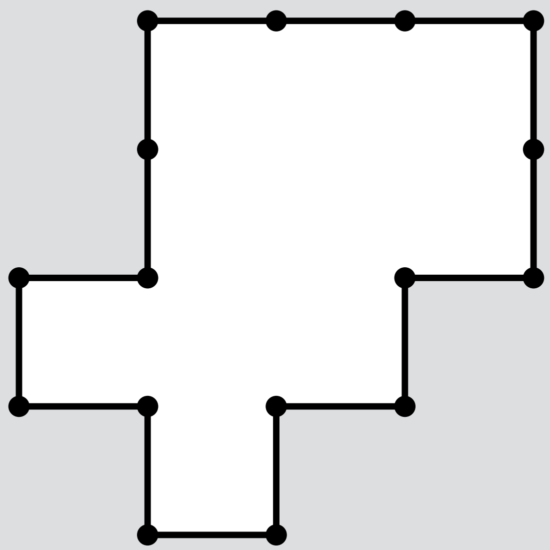
## Rectilinear simple polygons with unit edge lengths



# Boundary words

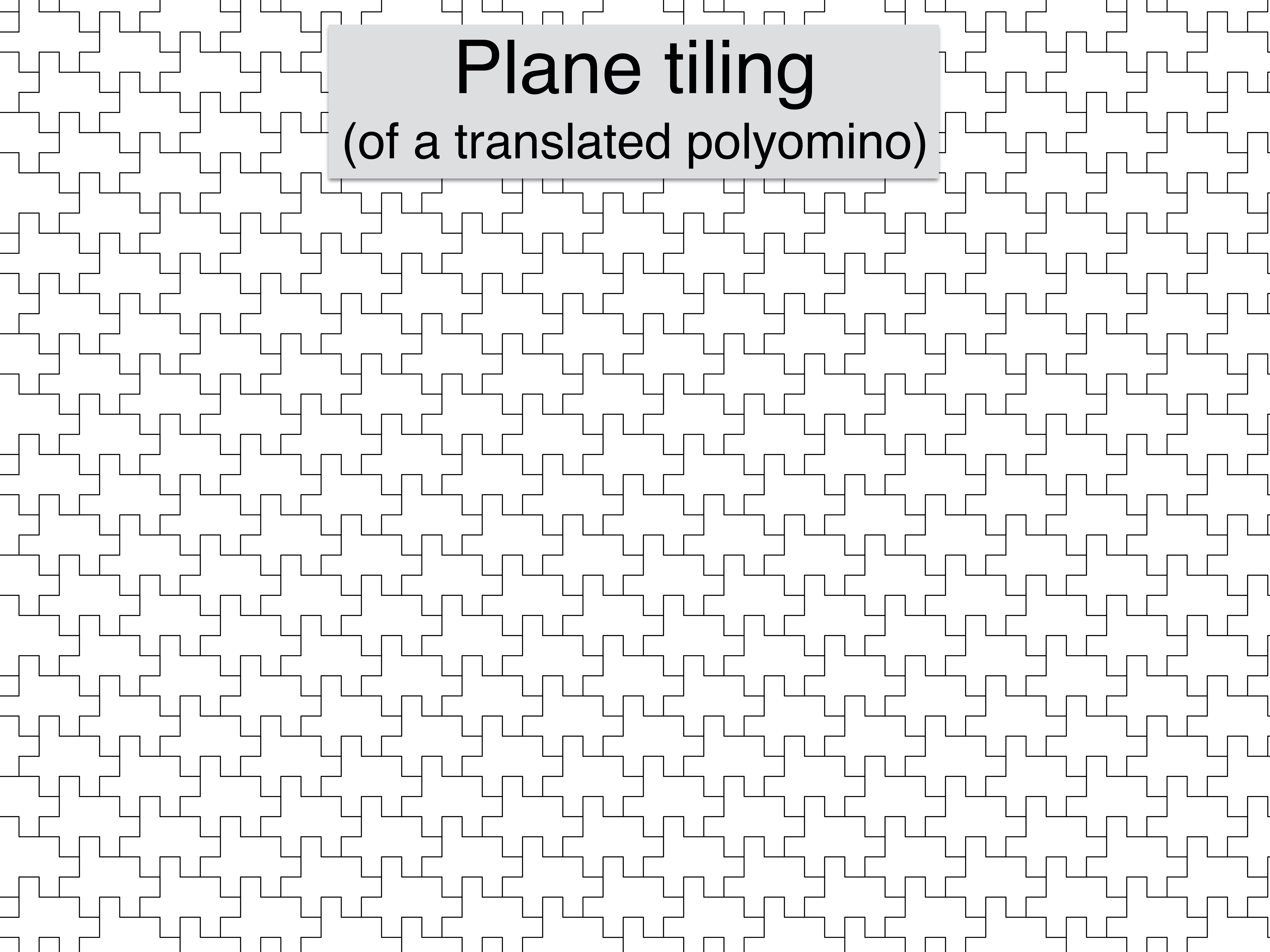


$uru^2rdr^2drd(ldlu)^2l$



$d(dl)^3uluru^2r^3$

# Plane tiling
## (of a translated polyomino)



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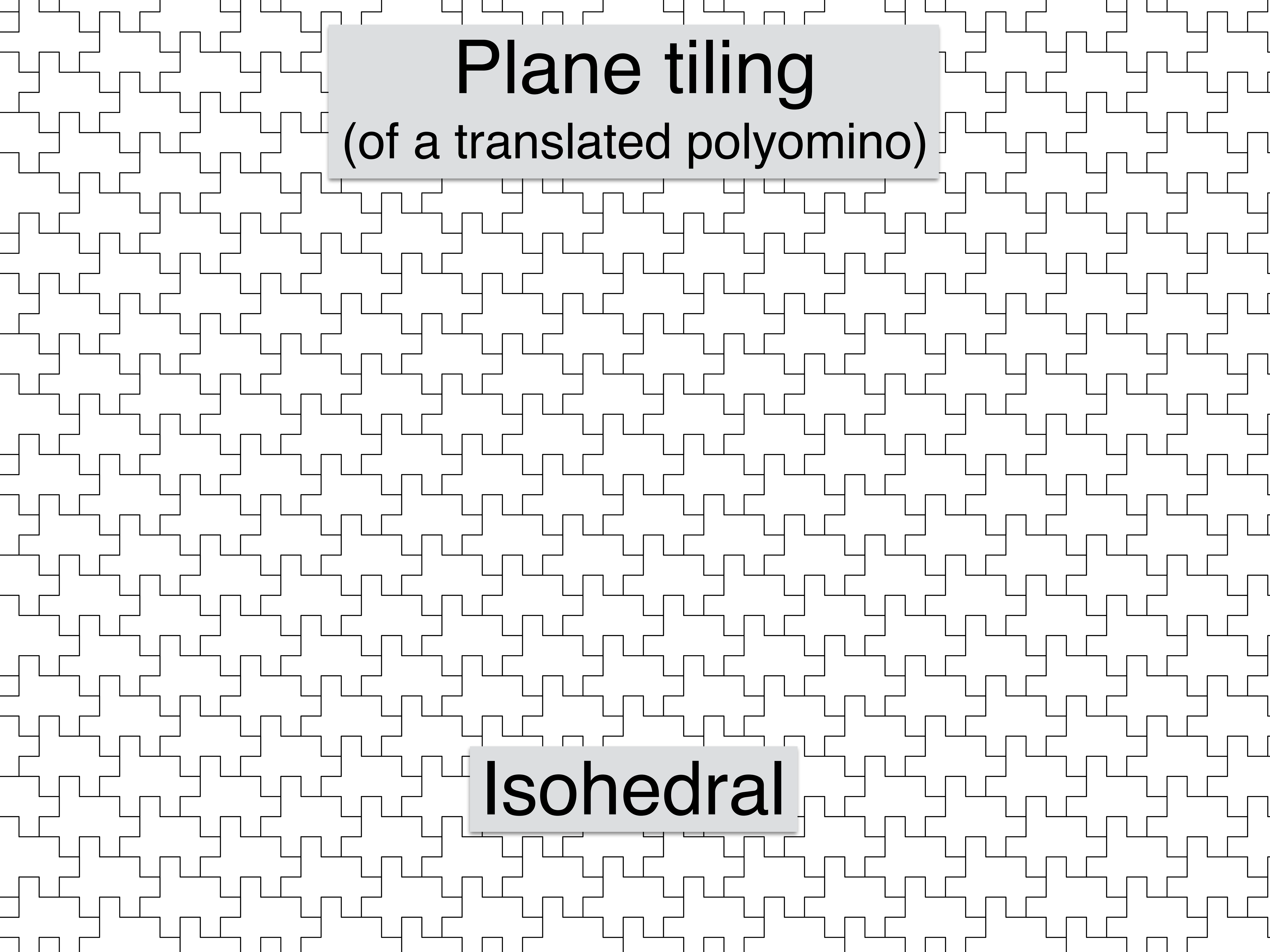

Isohedral

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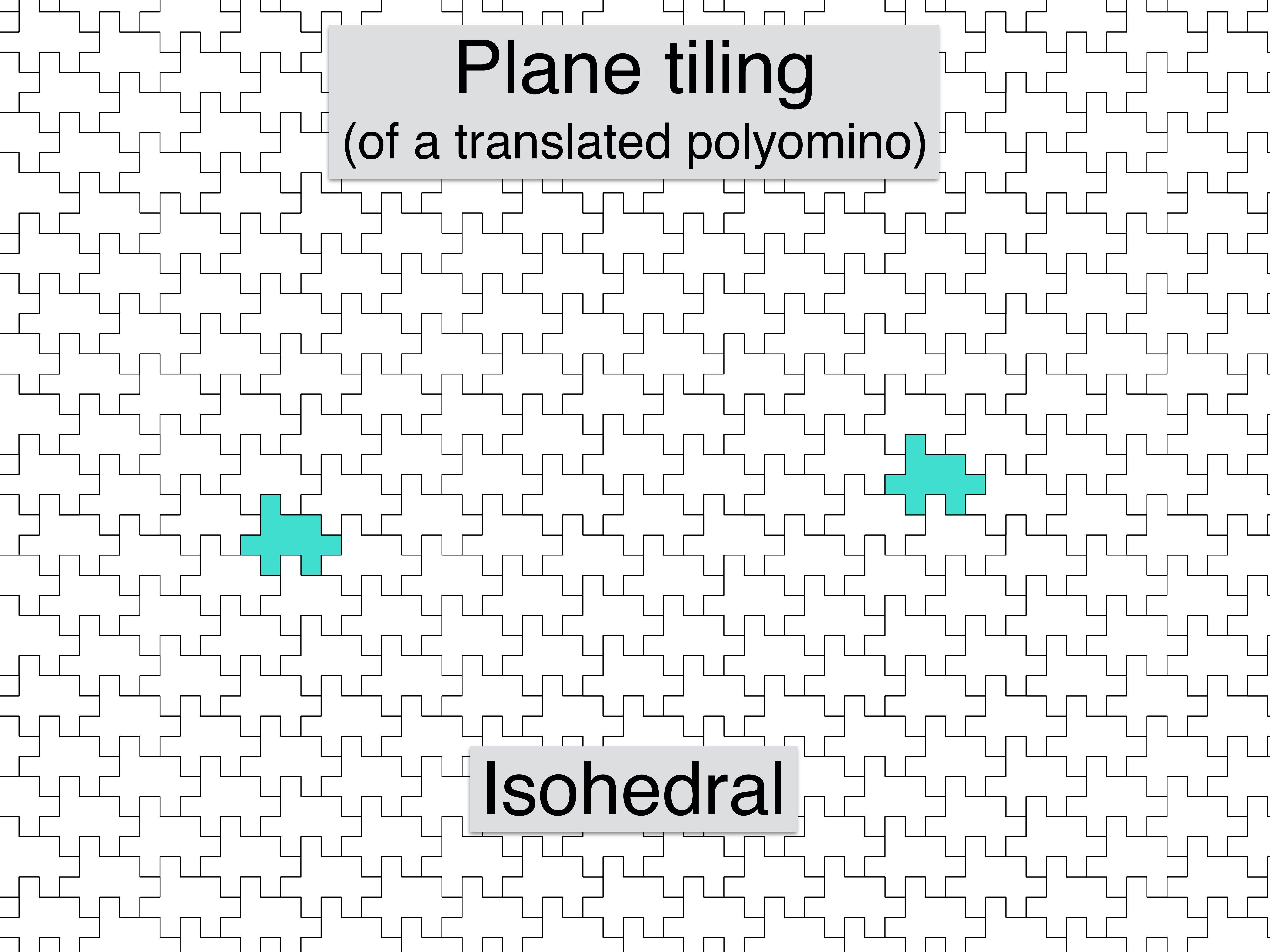

Isohedral

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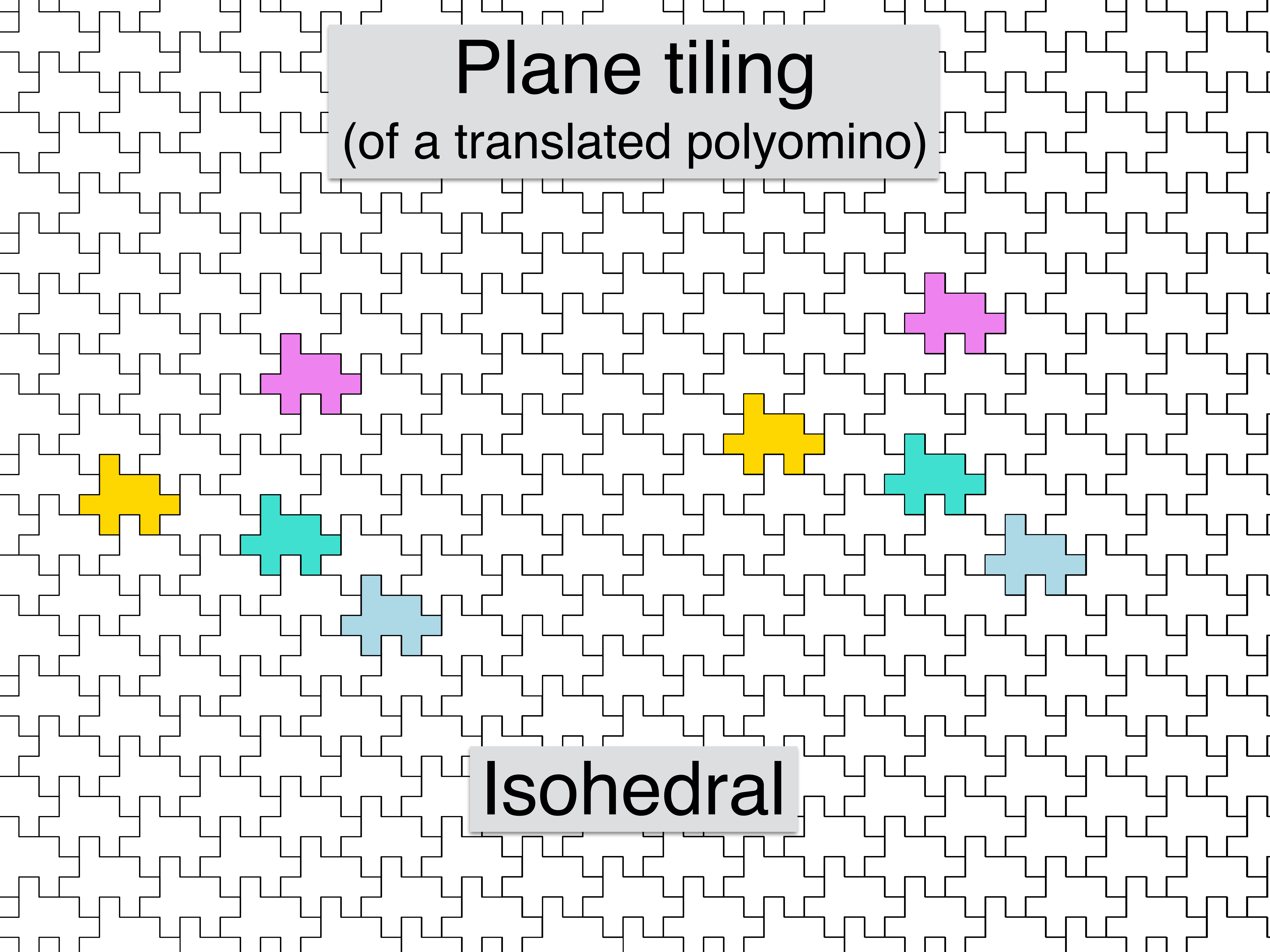
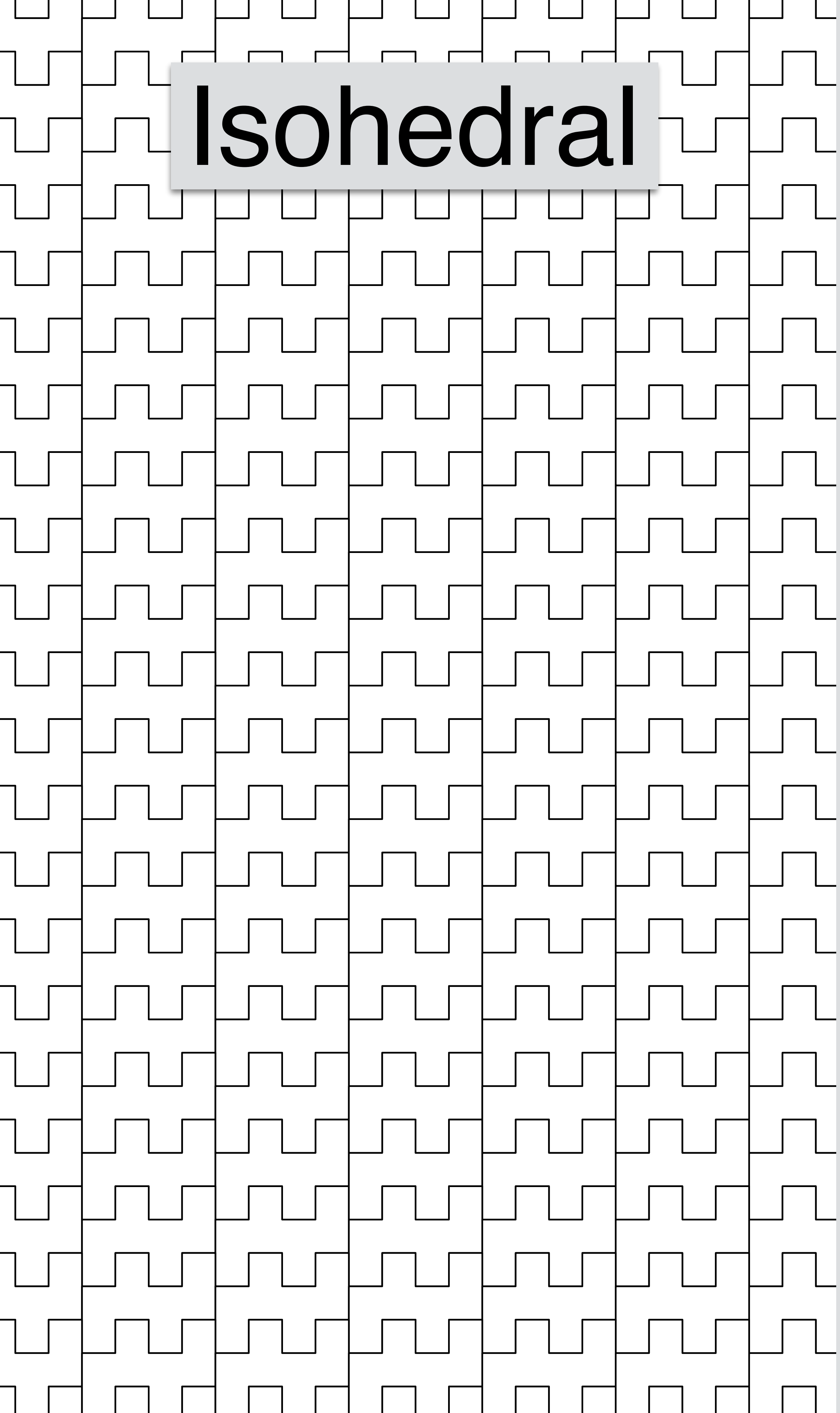
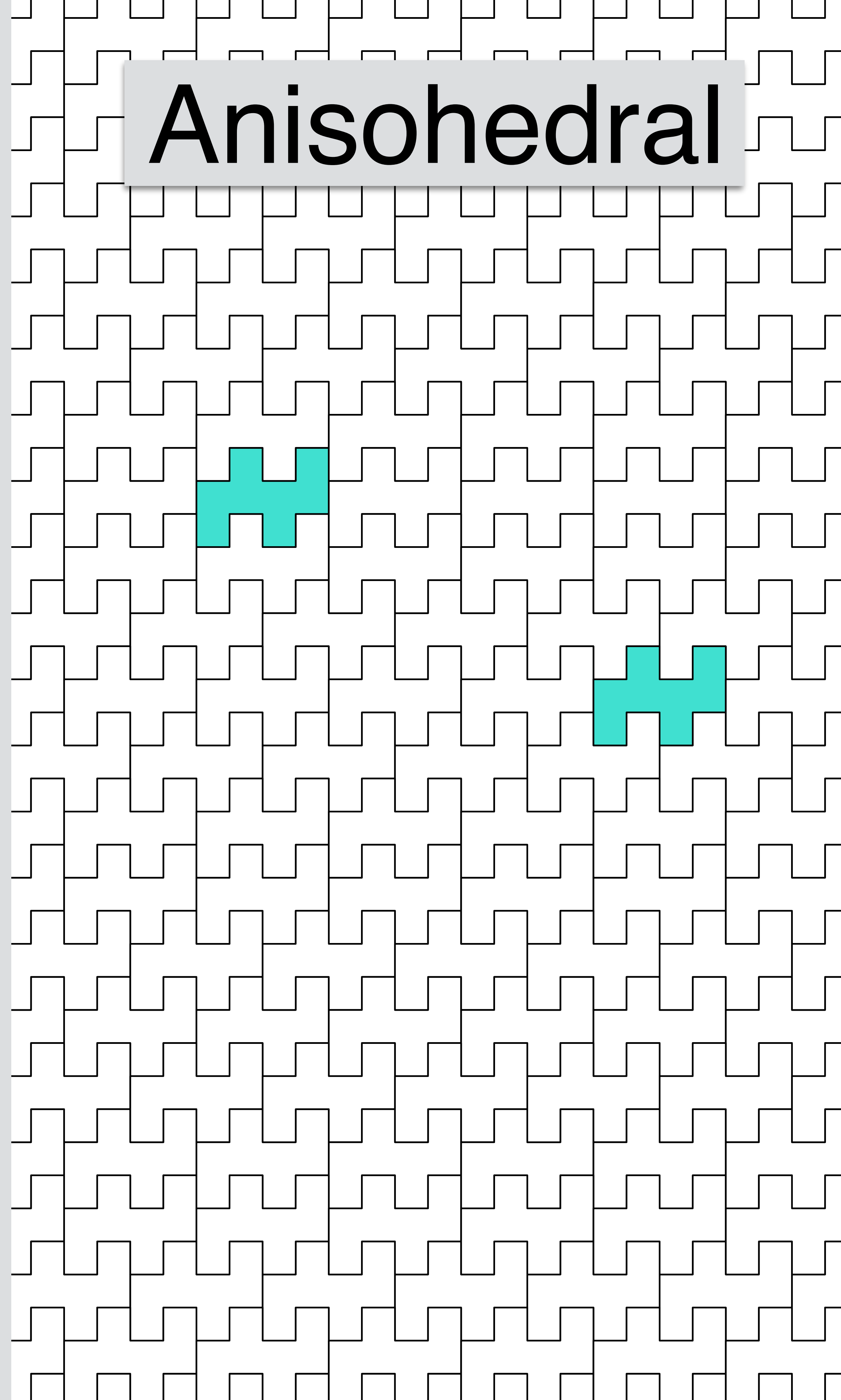


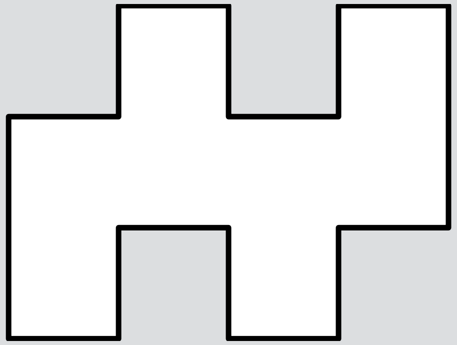
Isohedral


Isohedral

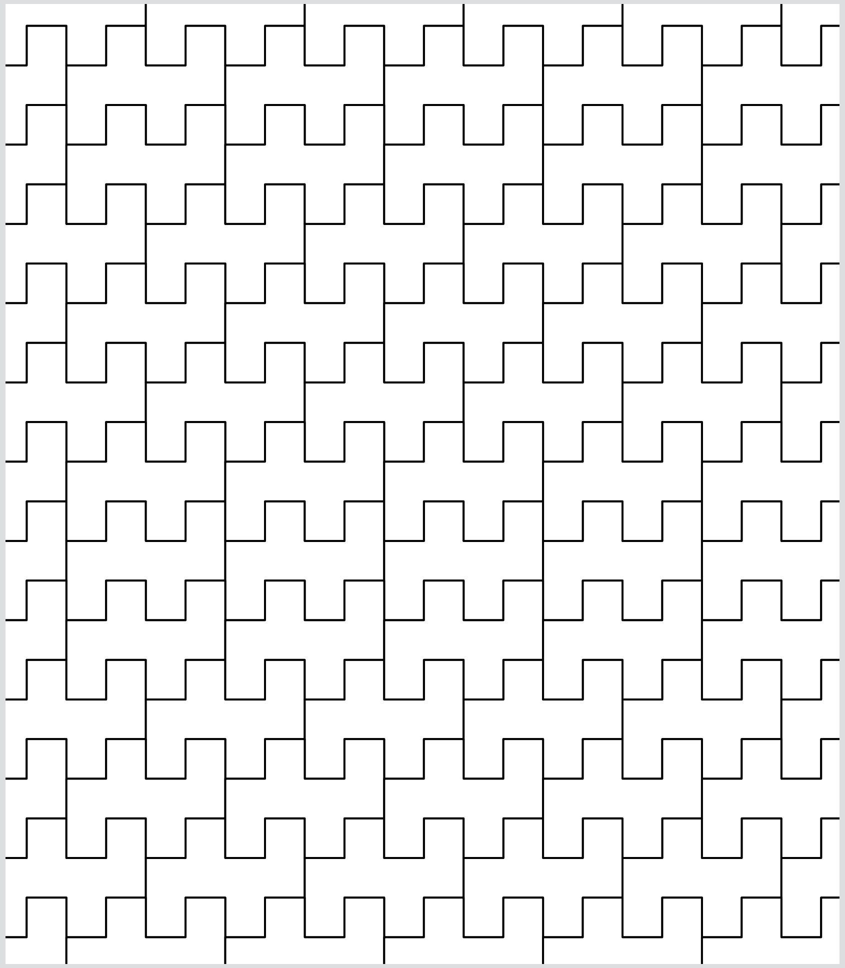

Anisohedral

[Wijshoff, van Leeuwen 1984], [Beauquier, Nivat 1991]:

For any polyomino P

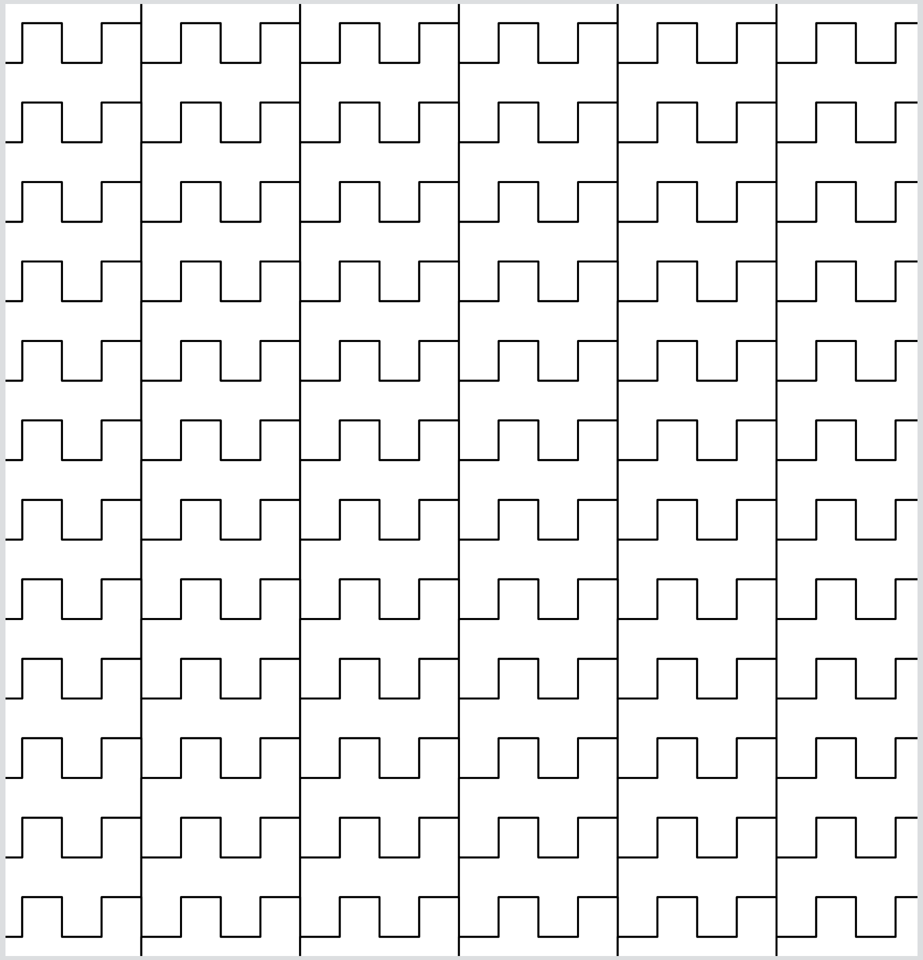


P has a plane tiling



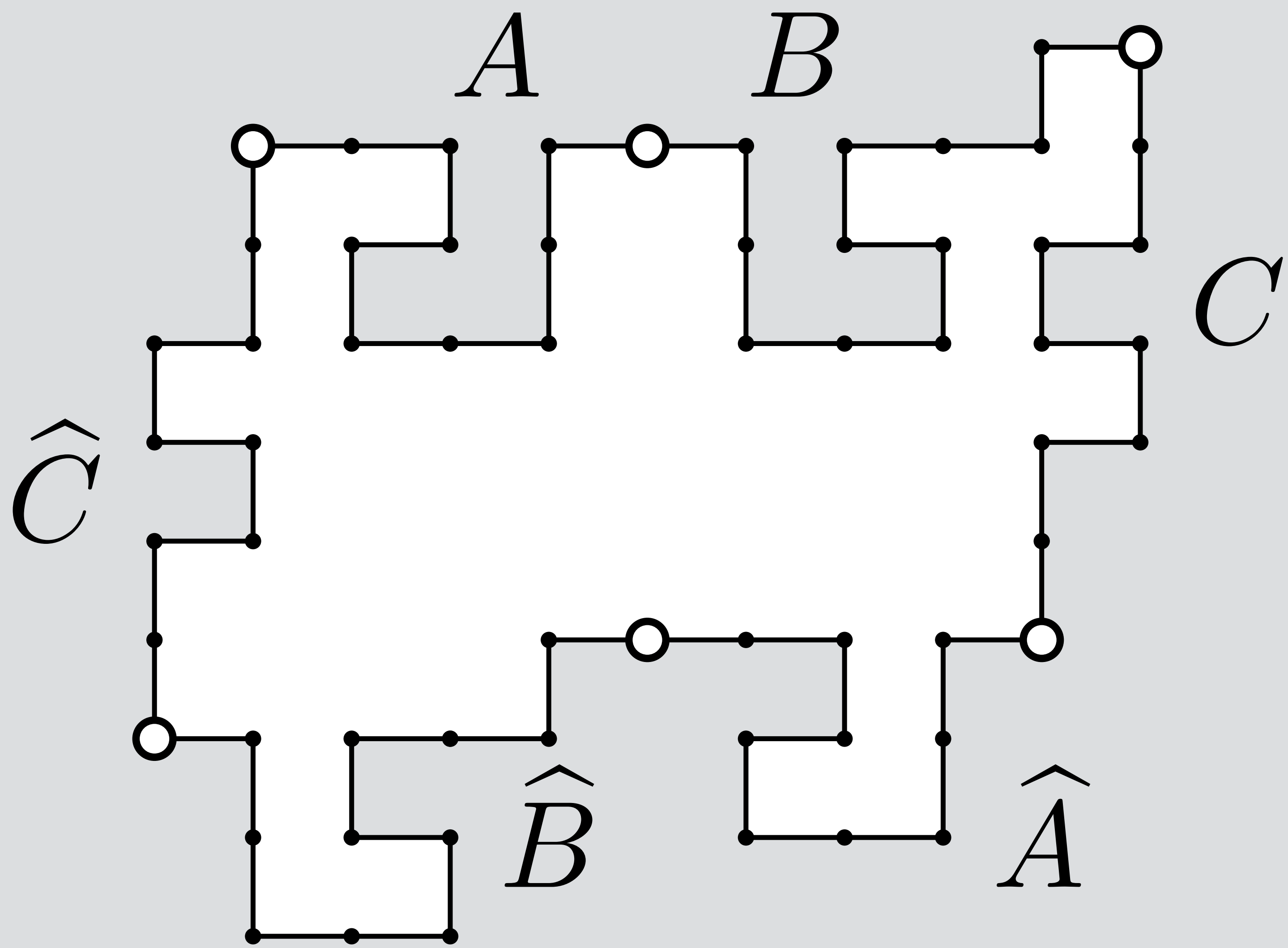
if and only if

P has an isohedral plane tiling



[Beauquier, Nivat 1991]:

A polyomino has an isohedral plane tiling if and only if the boundary word has factorization $ABC\widehat{A}\widehat{B}\widehat{C}$ :




where if $X = x_1 x_2 \ldots x_n$

then $\widehat{X} = \overline{x}_n \overline{x}_{n-1} \ldots \overline{x}_1$

with $\overline{u} = d \quad \overline{r} = l$

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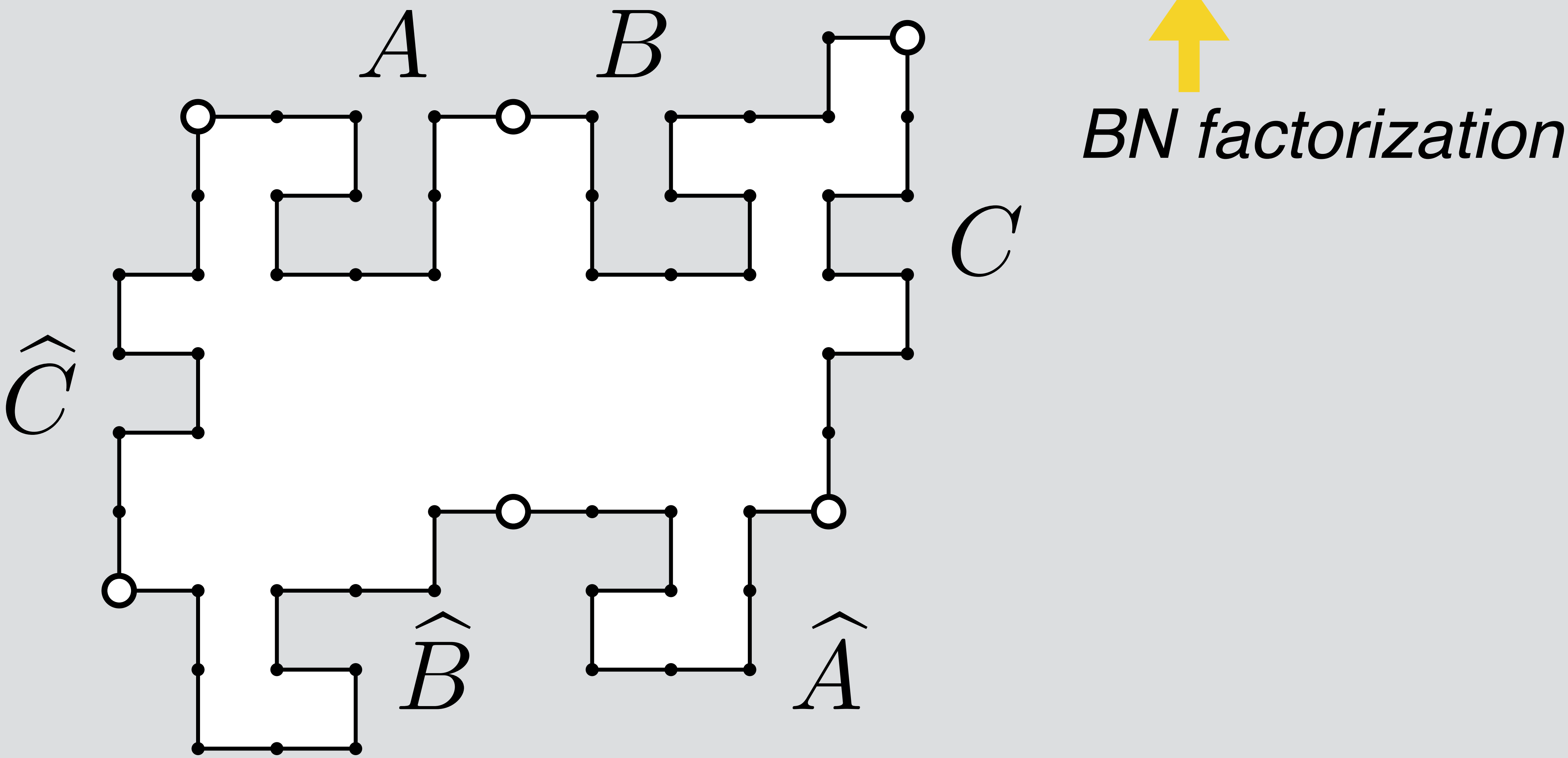


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# [Beauquier, Nivat 1991]:

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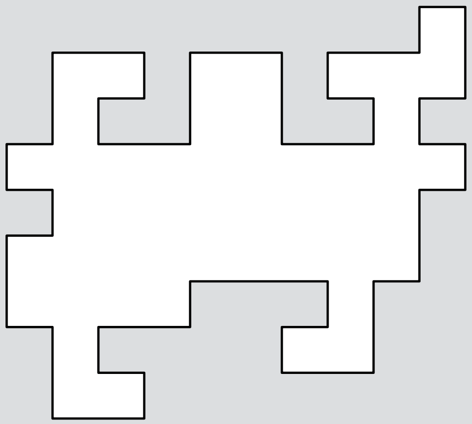
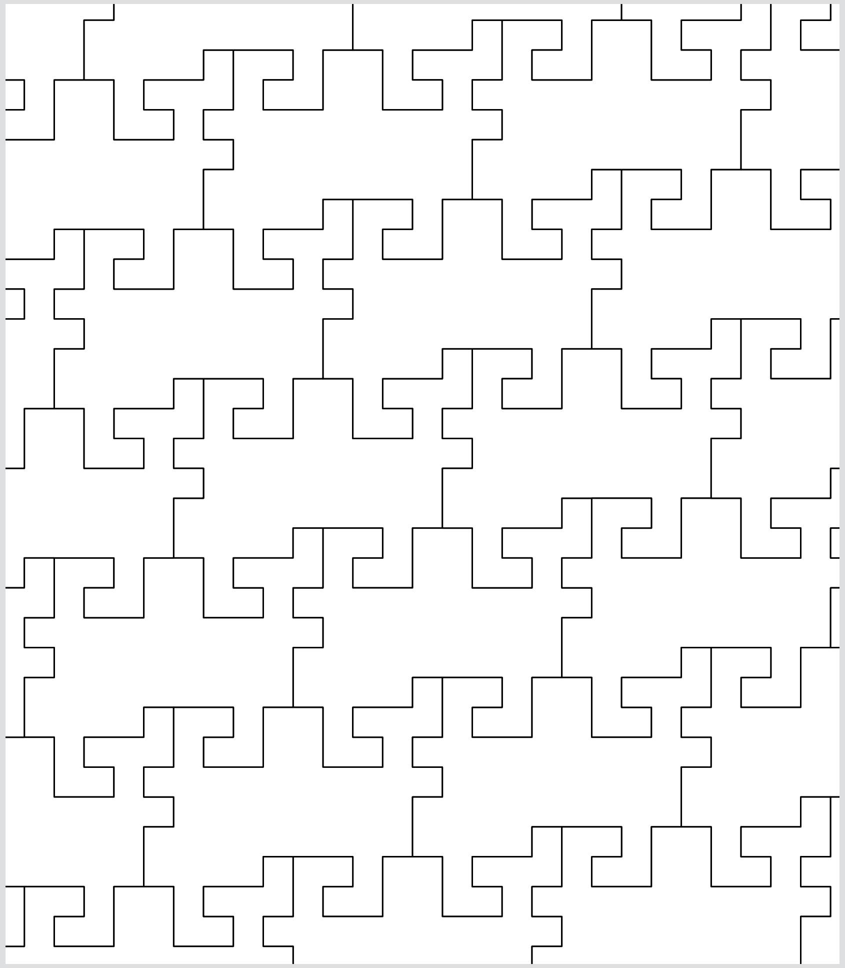
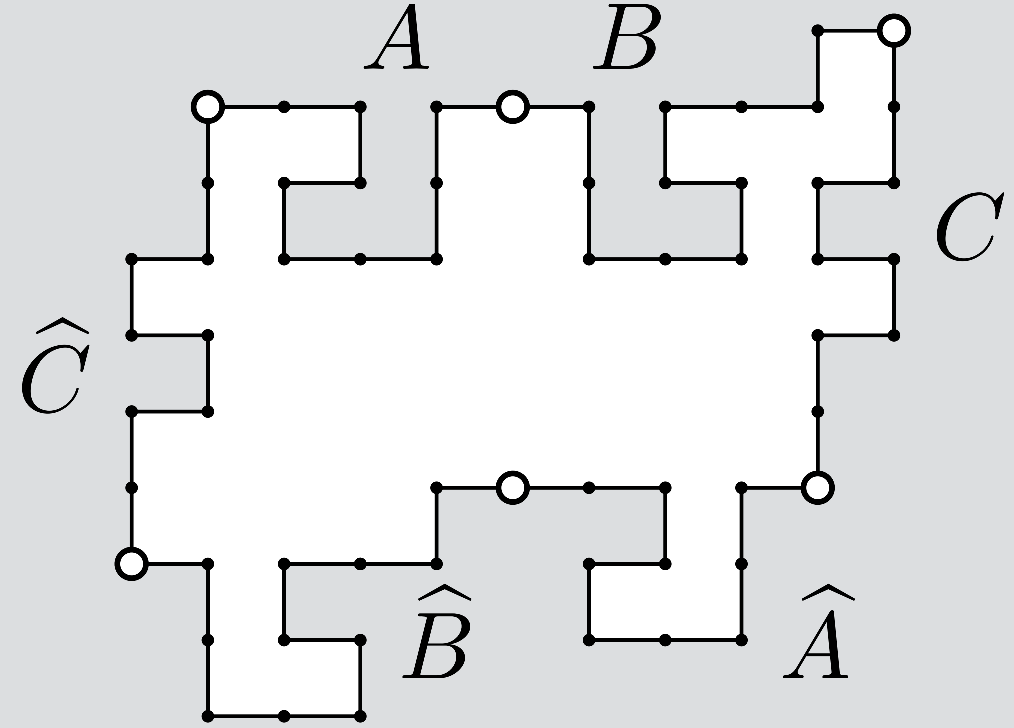


P has a plane tiling



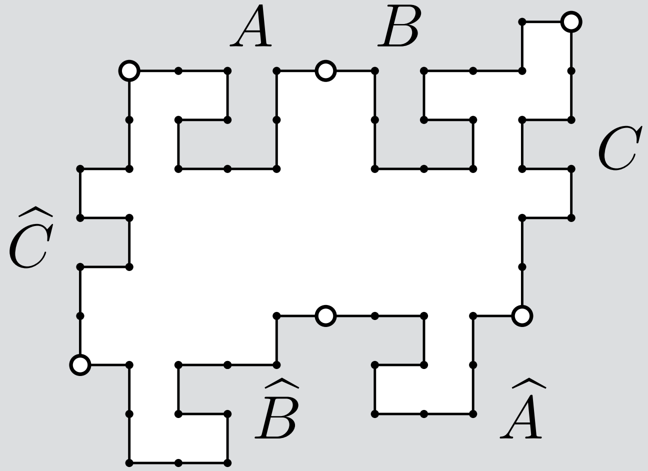
if and only if

Boundary word of P has BN factorization



# Testing for BN factorization

Given boundary word W with |W| = n, does W = $ABC\widehat{A}\widehat{B}\widehat{C}$?




- [Gambini, Vuillon 2007]: $O(n^2)$
- [Provençal 2008]: $O(n*\log^3(n))$
- [Brlek, Provençal, Fédou 2009]: O(n) in two special cases.

This work: O(n) algorithm for all inputs.

# The algorithm

# Admissible factors

Lemma #0: Every factor in a BN factorization is an admissible factor.

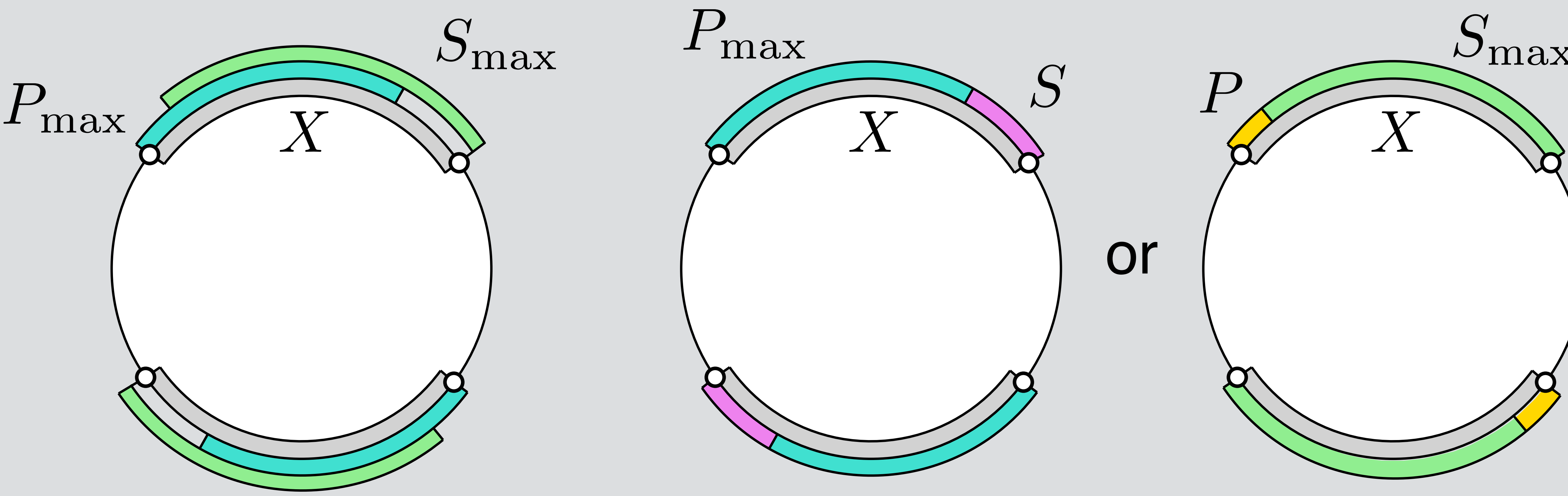
Lemma #1 ([Brlek et al. 2009]): the O(n)-sized set of all admissible factors can be computed in O(n) time.

Lemma #2: BN factorization if and only if consecutive admissible factors A,B,C with |ABC| = n/2.

Lemma #3: X has factorization into two admissible factors if and only if $X = P_{max}S$ or $X = PS_{max}$ with:

1. $P_{max}$ the longest prefix admissible factor of X, or
2. $S_{max}$ the longest suffix admissible factor of X.

and P, S admissible factors.

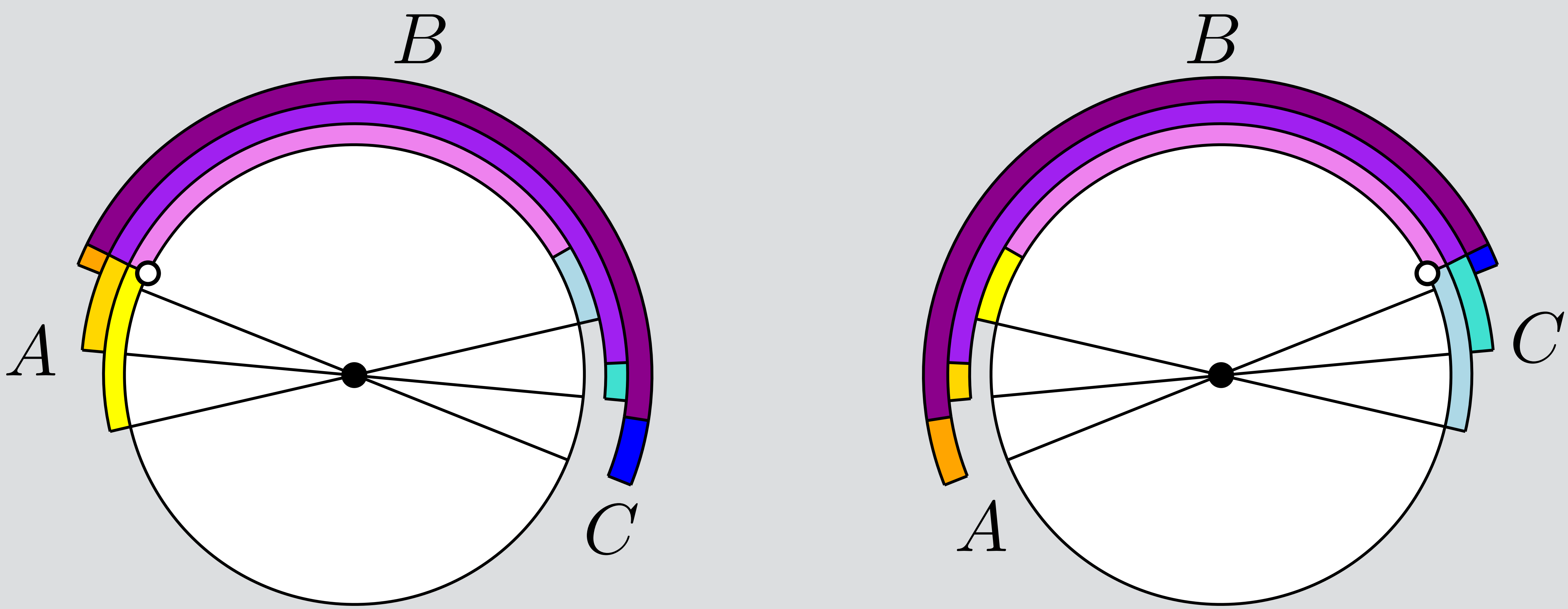



Proof follows that of similar result by [Galil, Seiferas 1978]

# Finding consecutive A,B,C with |ABC| = n/2.

1. For each A, search for longest B such that |AB| ≤ n/2, check whether factor C with |ABC| = n/2 is admissible.

2. For each C, search for longest B such that |BC| ≤ n/2, check whether factor A with |ABC| = n/2 is admissible.



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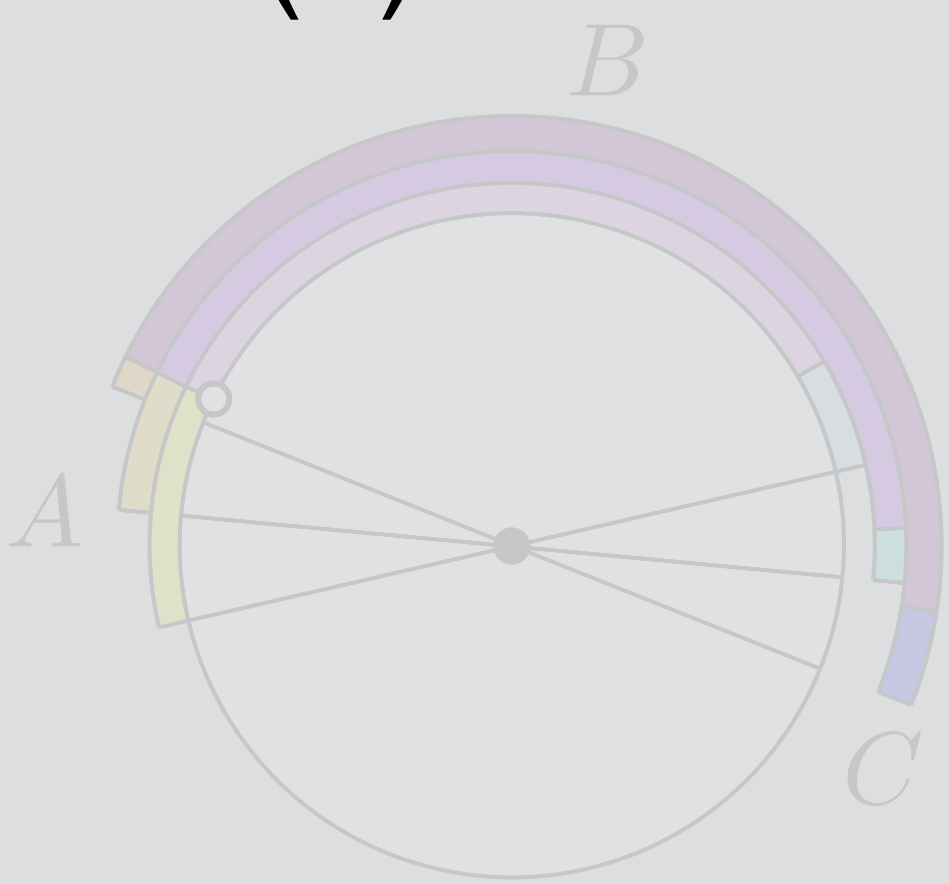
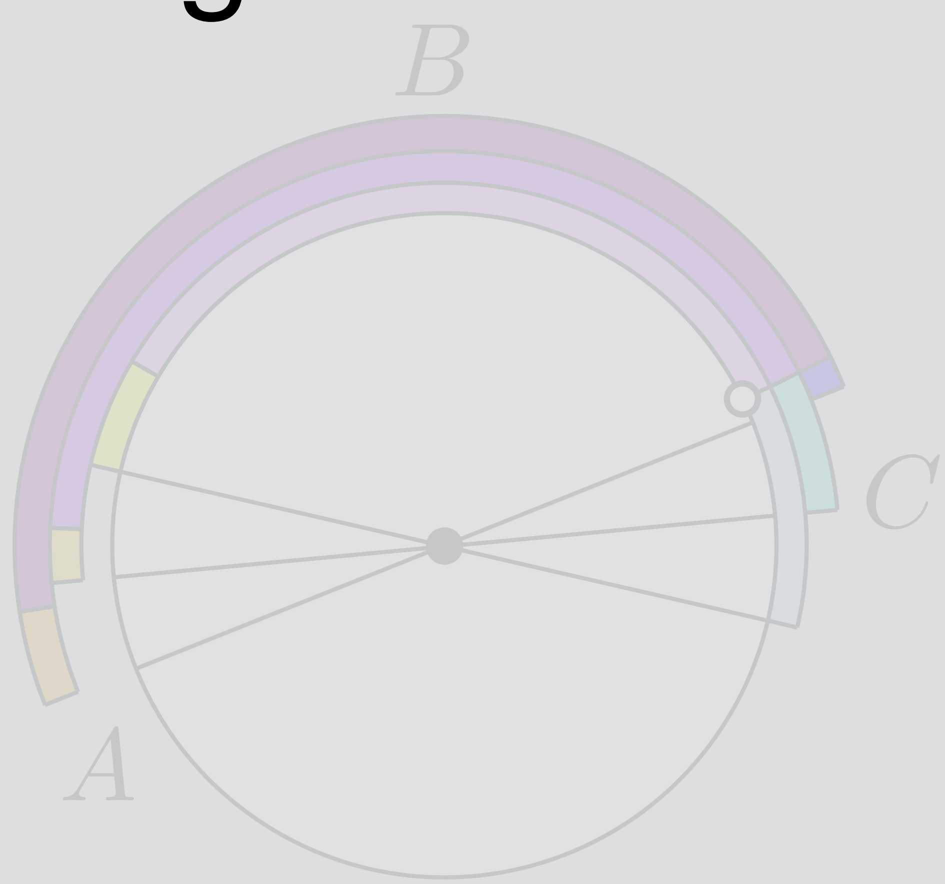
O(n) time using two-finger scans.



# Algorithm

1. Compute all admissible factors.

2. Sort admissible factors starting and ending at each letter.

3. Two-finger scans to search for A,B,C with $|ABC| = n/2$.

O(n)-time algorithm

# Enumeration

1. BN factorizations = isohedral tilings.

2. Algorithm can be modified to enumerate all k factorizations in $O(n+k)$ time.

3. This work also proves $k = O(n)$, so $O(n)$-time enumeration of isohedral tilings.

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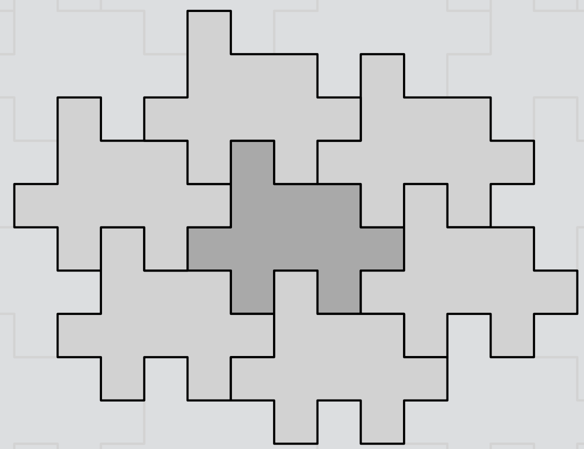

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