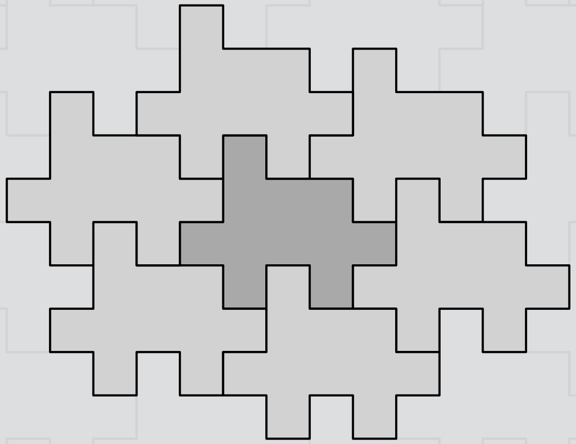
An Optimal Algorithm for Tiling the Plane with a Translated Polyomino



Andrew Winslow

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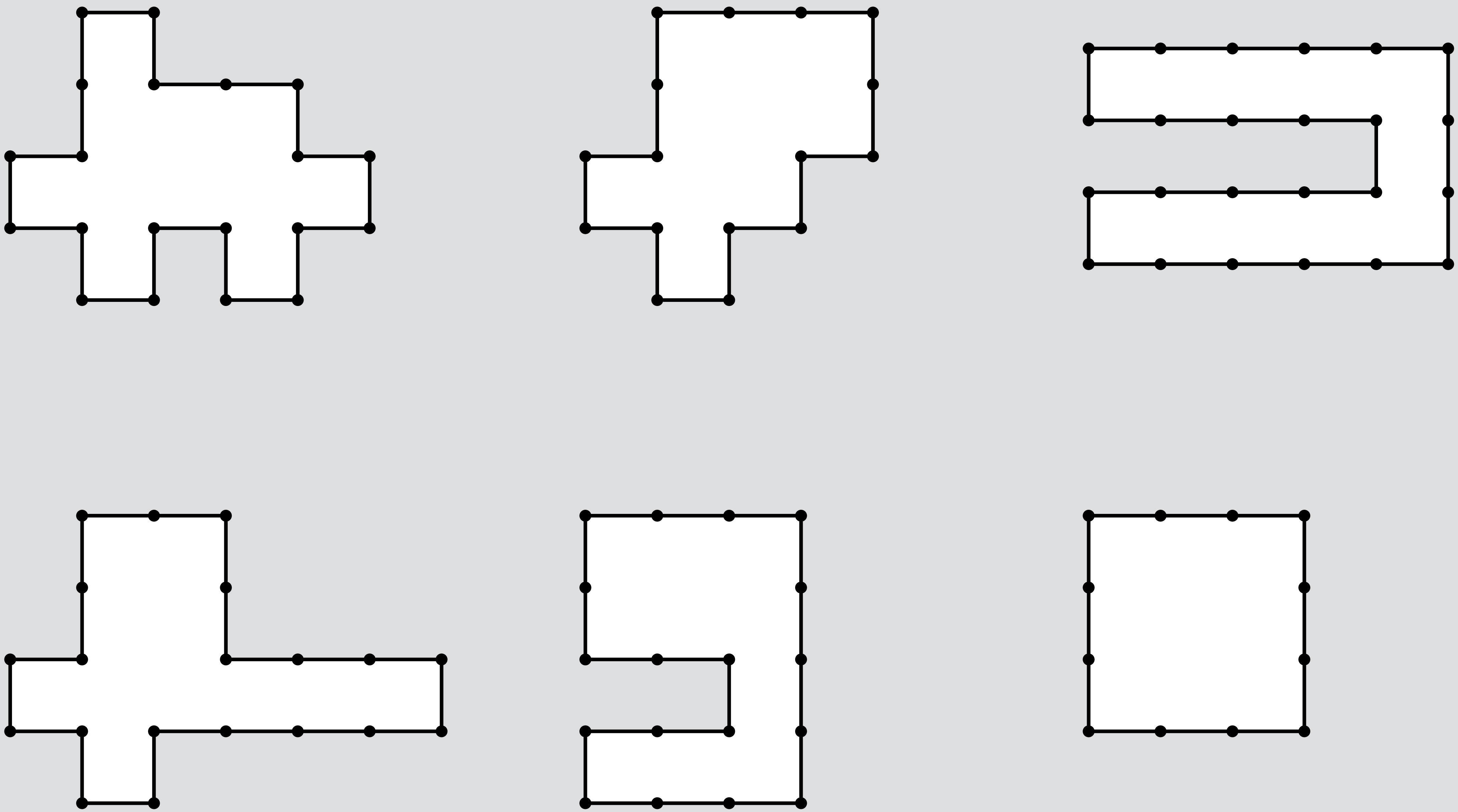
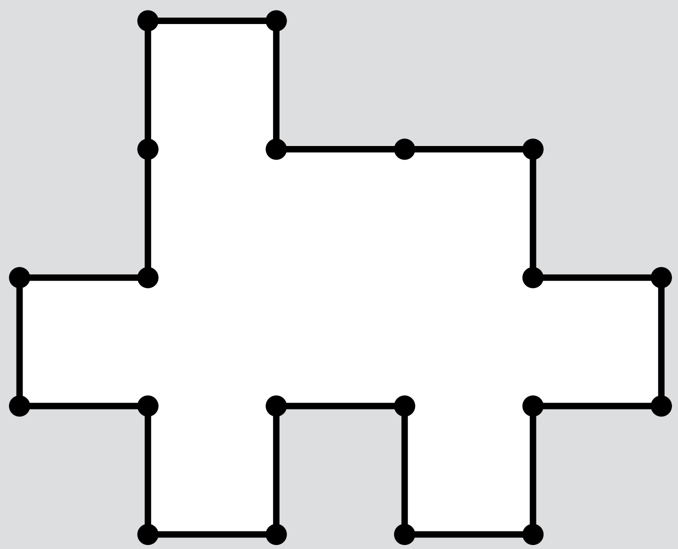
Polyominoes

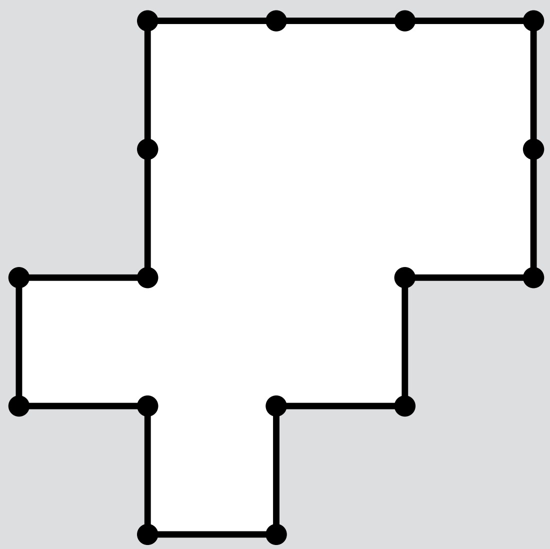
Rectilinear simple polygons with unit edge lengths



Boundary words

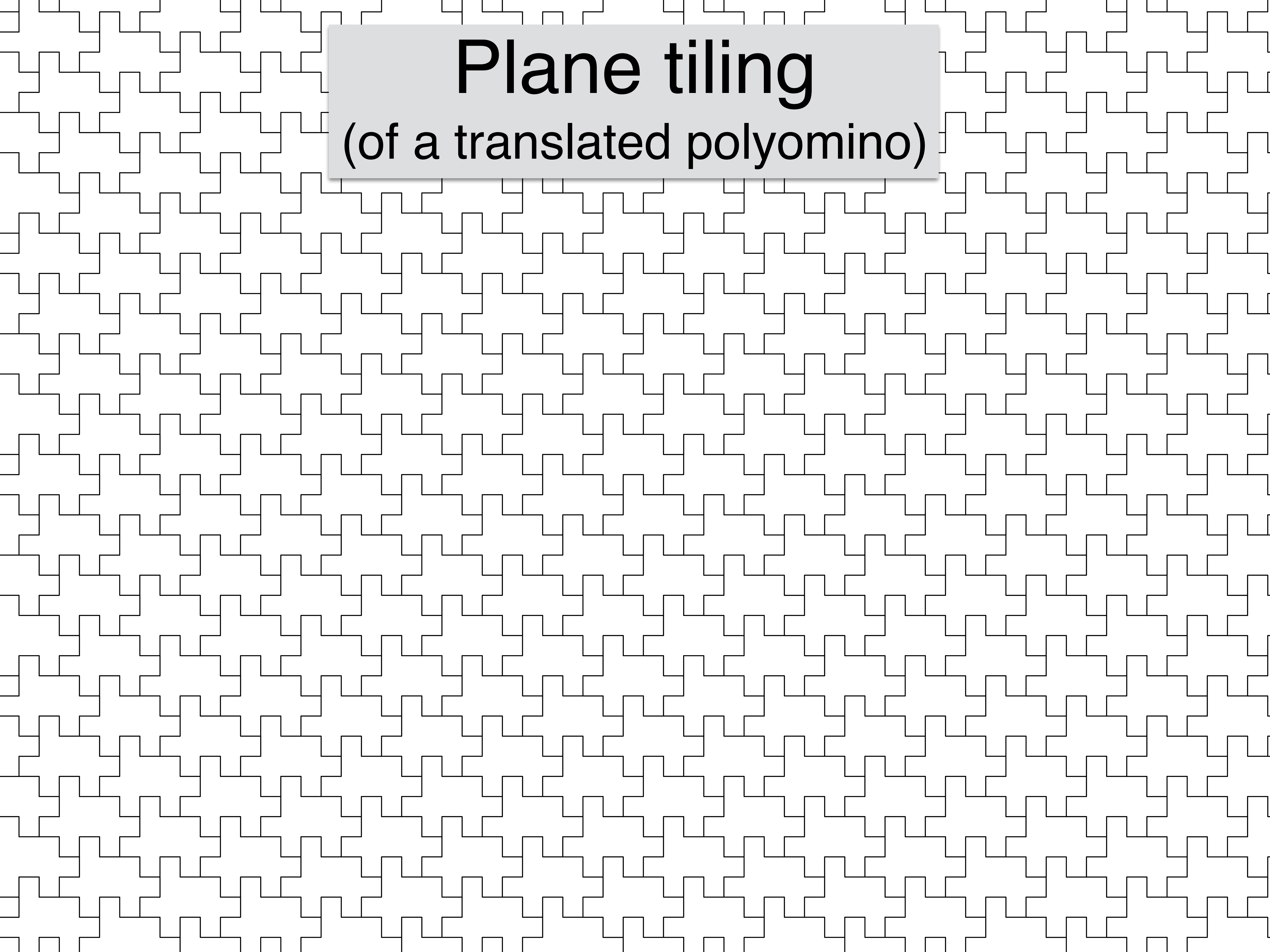


$$uru^2rdr^2drd(ldlu)^2l$$

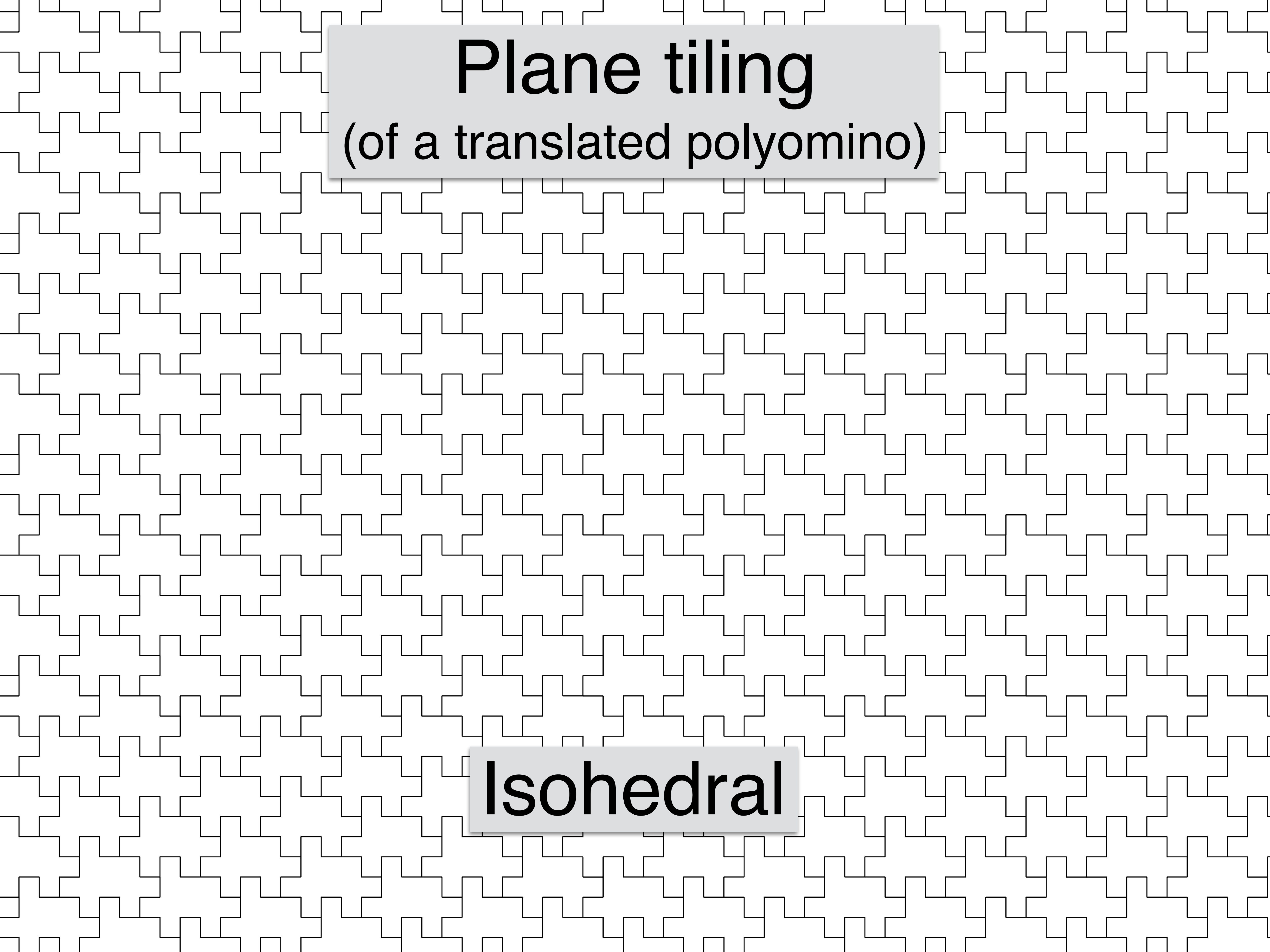


$$d(dl)^3uluru^2r^3$$

Plane tiling
(of a translated polyomino)



Plane tiling
(of a translated polyomino)



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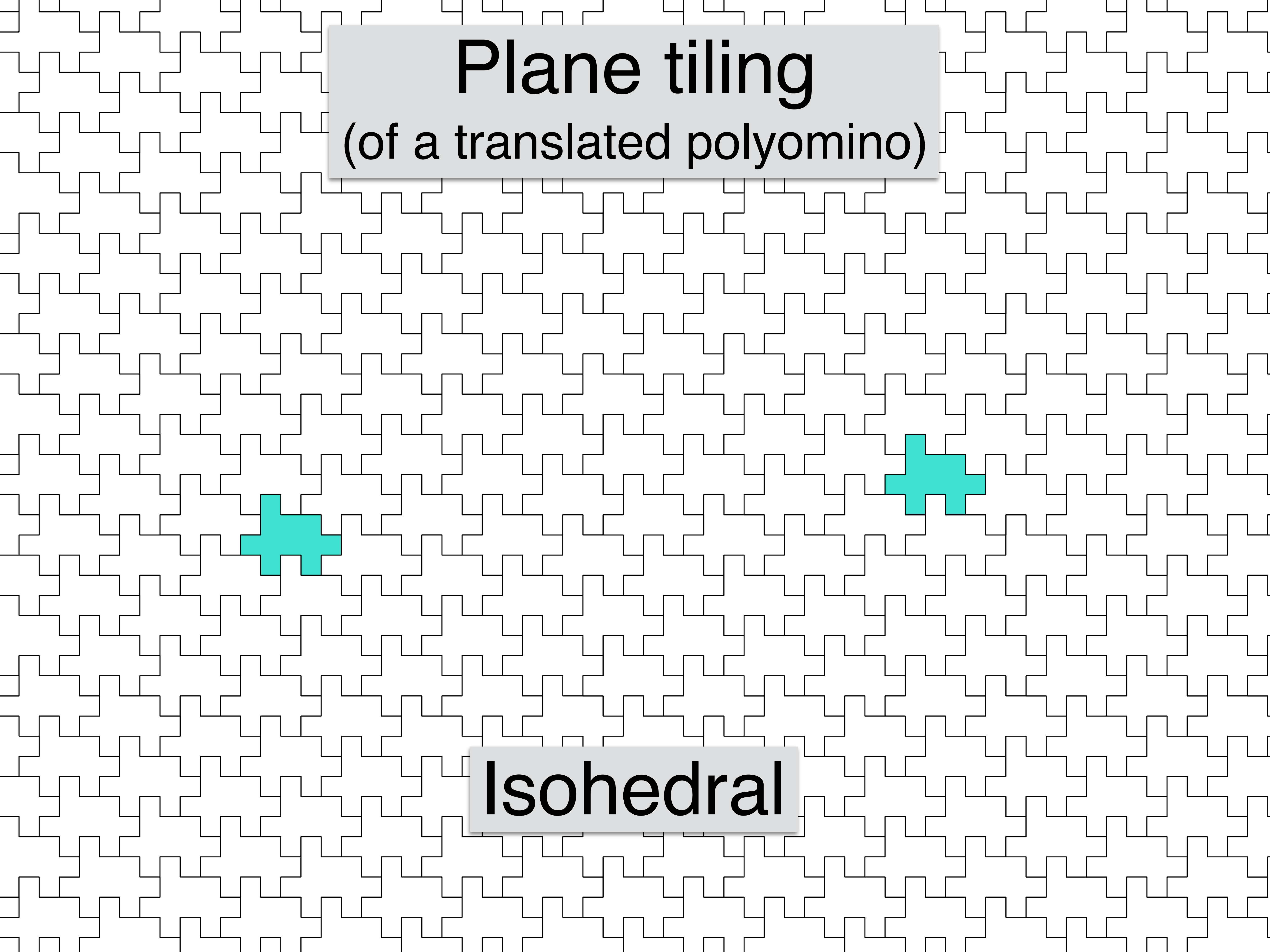

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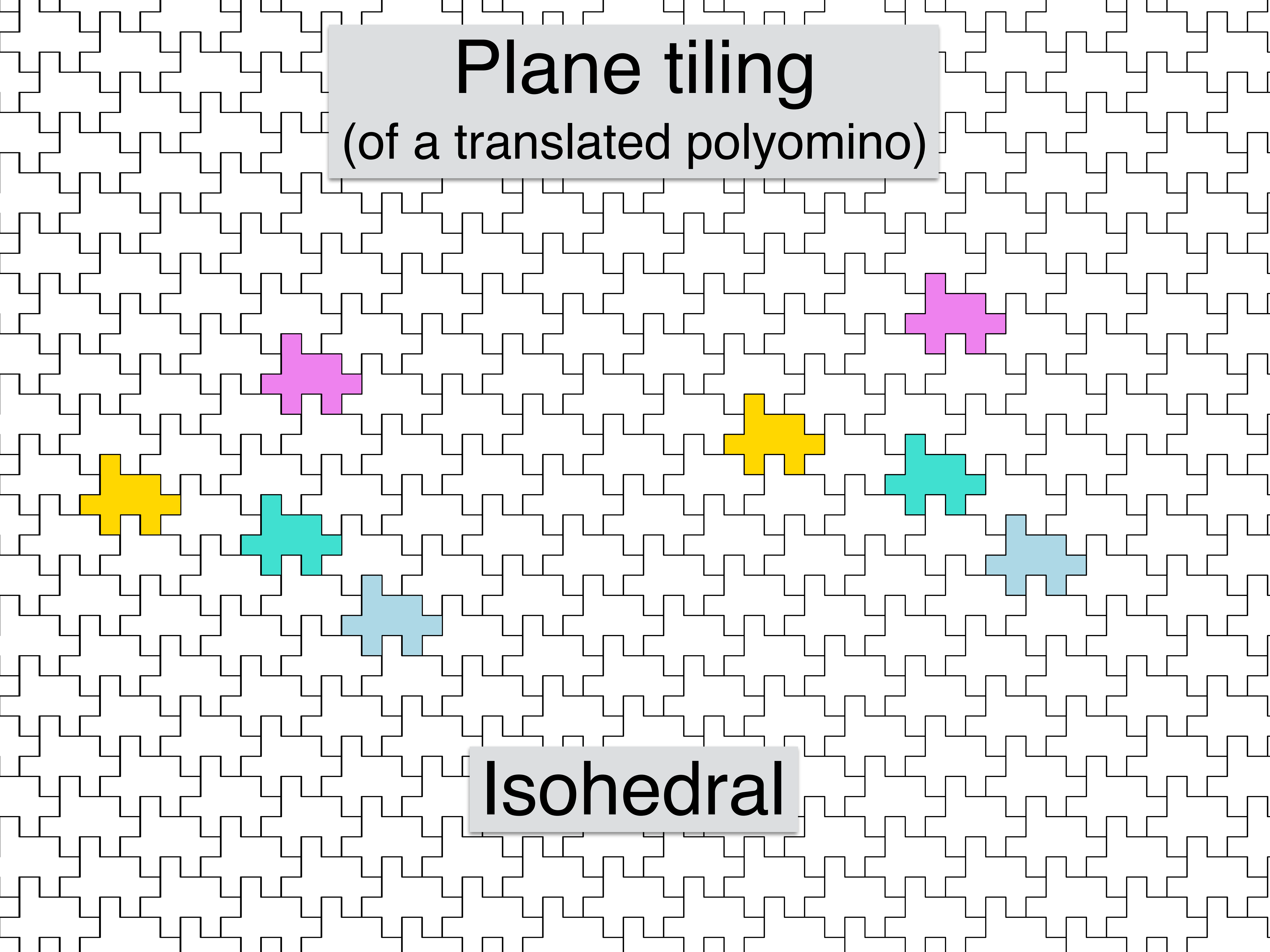
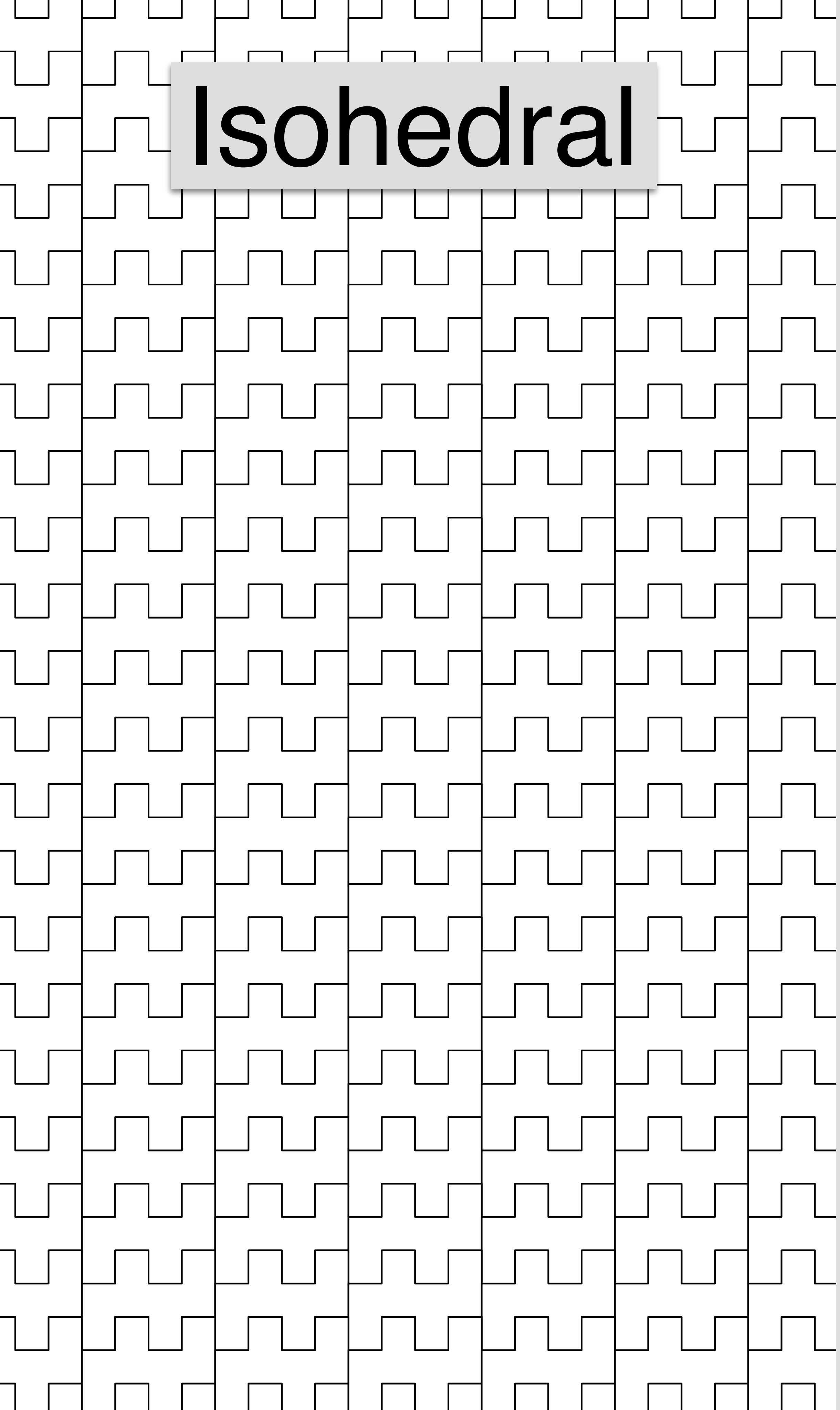
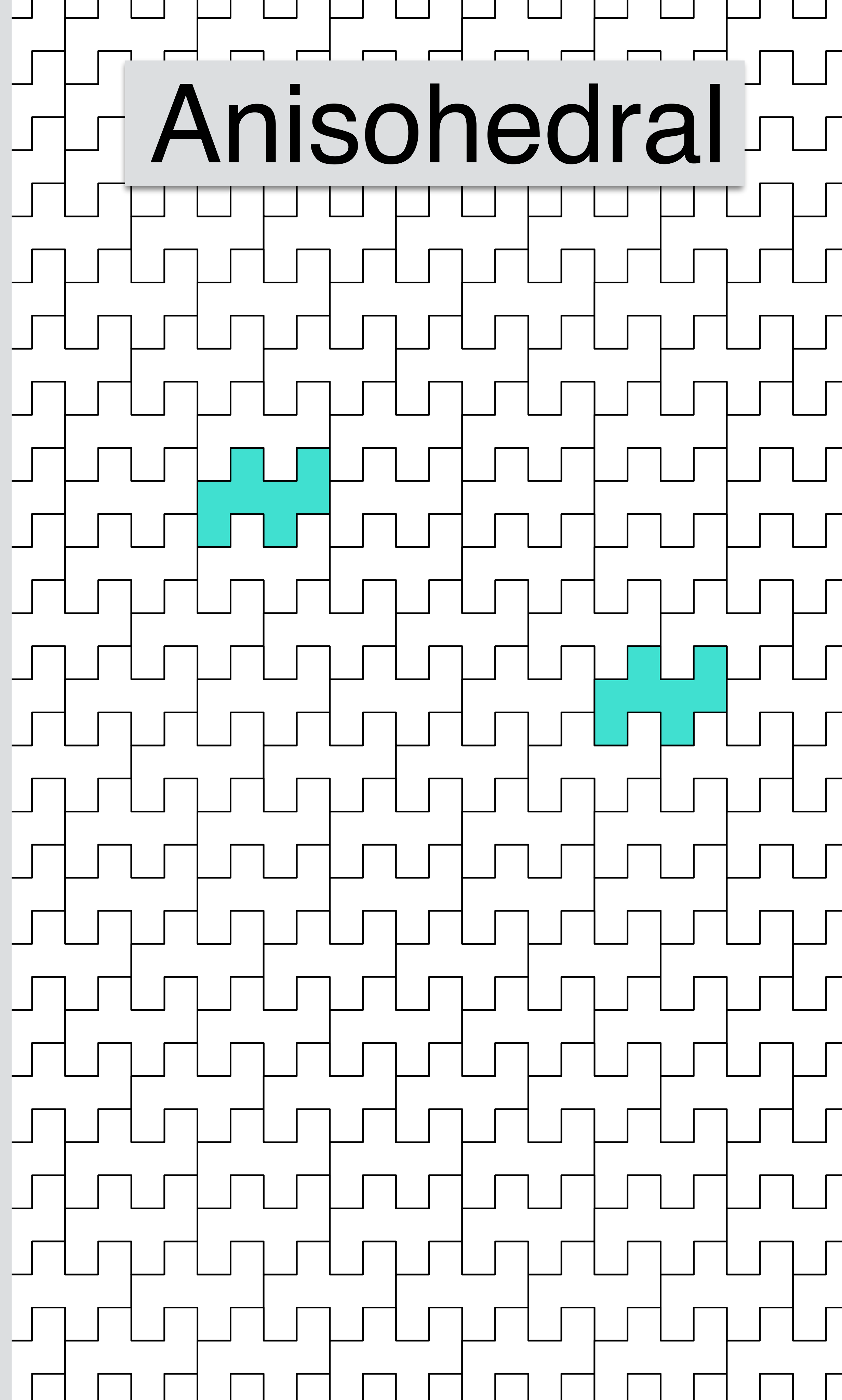
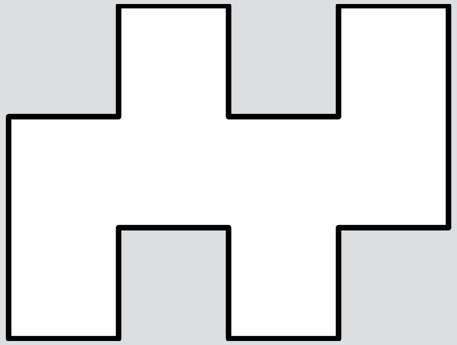



Isohedral

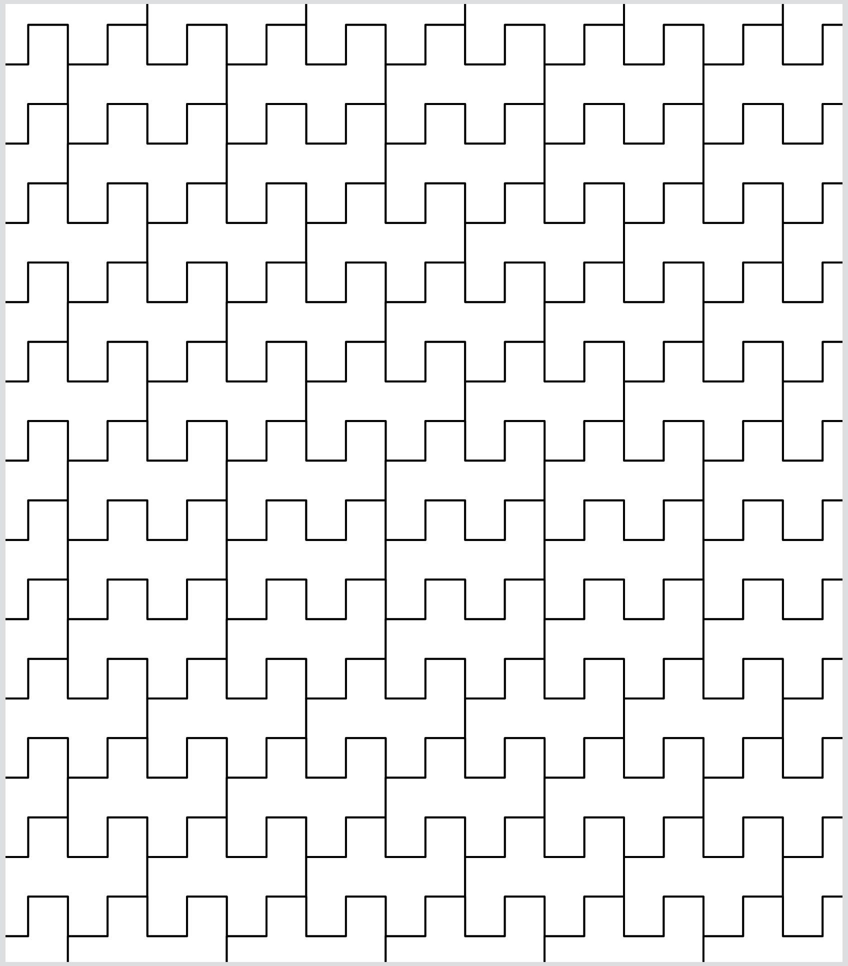

Anisohedral

[Wijshoff, van Leeuwen 1984], [Beauquier, Nivat 1991]:

For any polyomino P

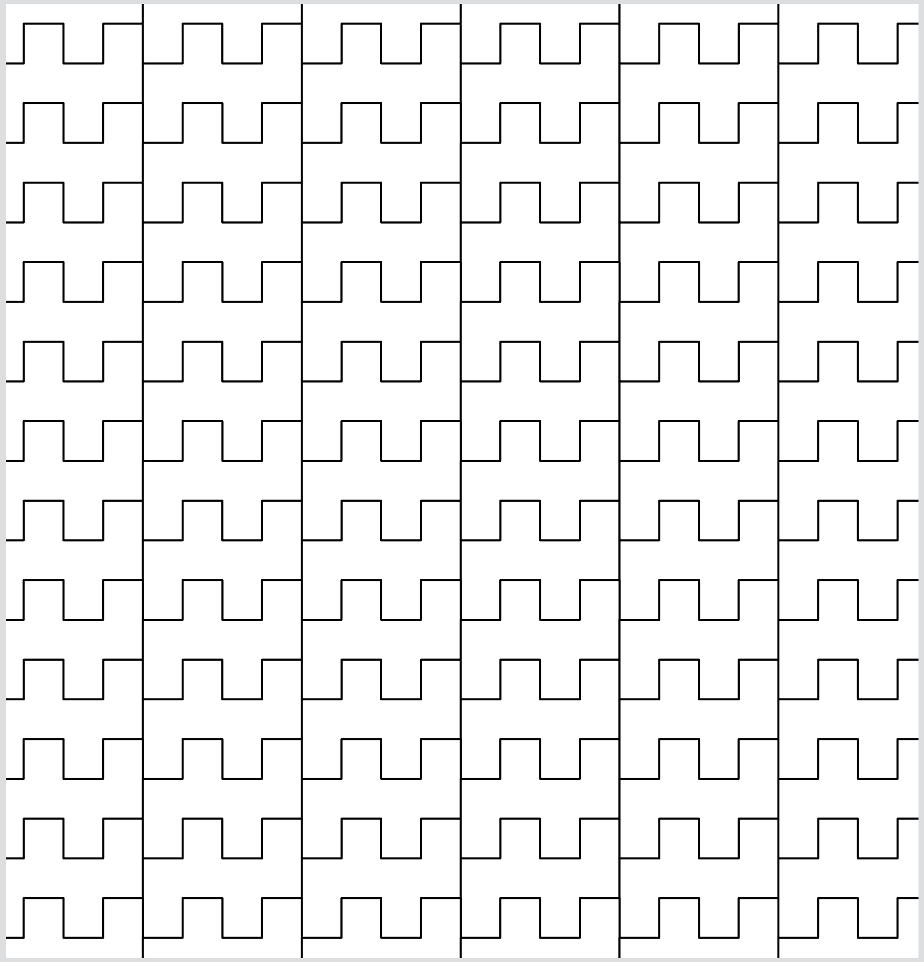


P has a plane tiling



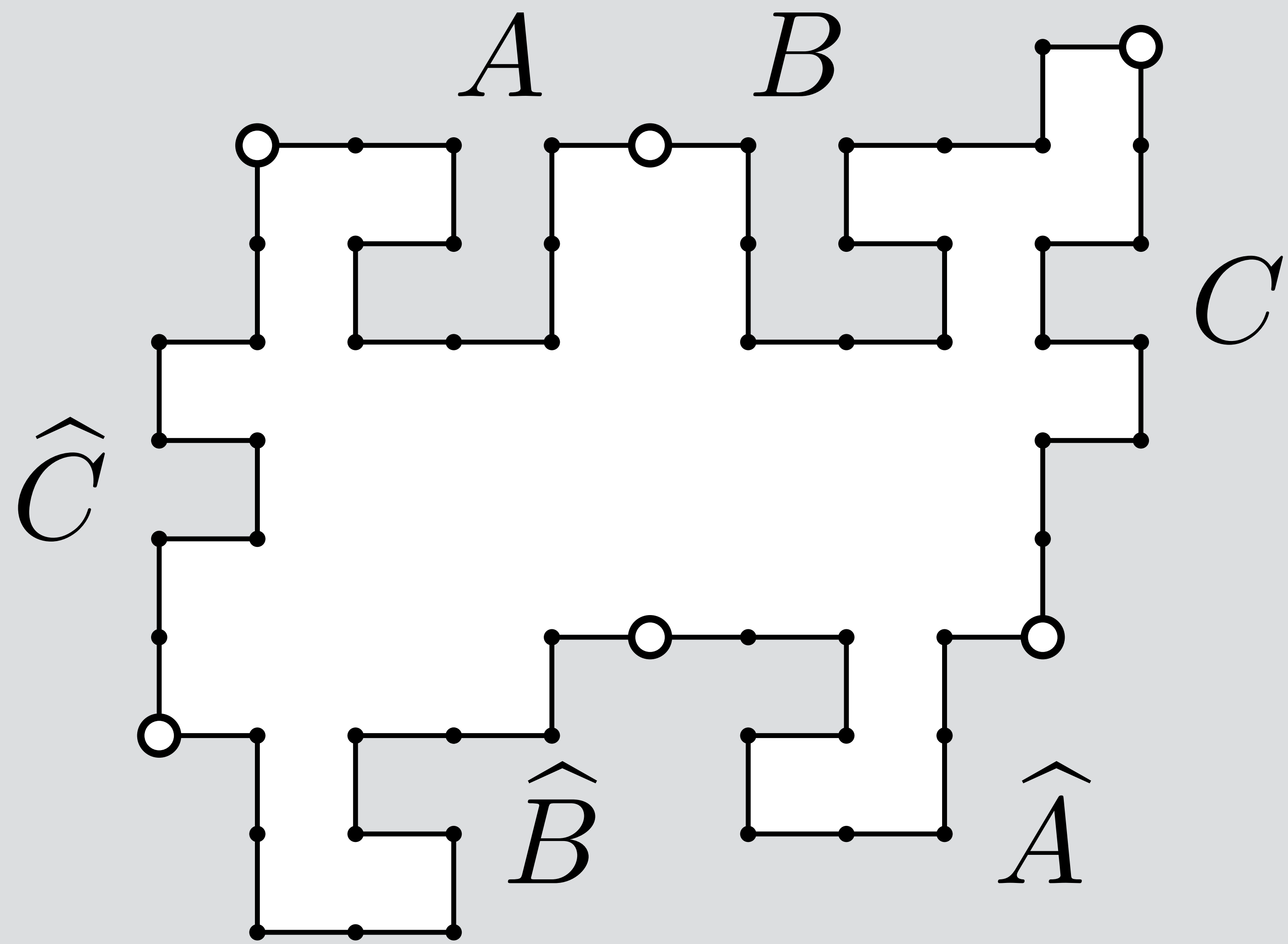
if and only if

P has an isohedral plane tiling



[Beauquier, Nivat 1991]:

A polyomino has an isohedral plane tiling if and only if the boundary word has factorization $ABC\widehat{A}\widehat{B}\widehat{C}$:



where if $X = x_1 x_2 \ldots x_n$
then $\widehat{X} = \overline{x}_n \overline{x}_{n-1} \ldots \overline{x}_1$
with $\overline{u} = d \quad \overline{r} = l$
$\overline{d} = u \quad \overline{l} = r$

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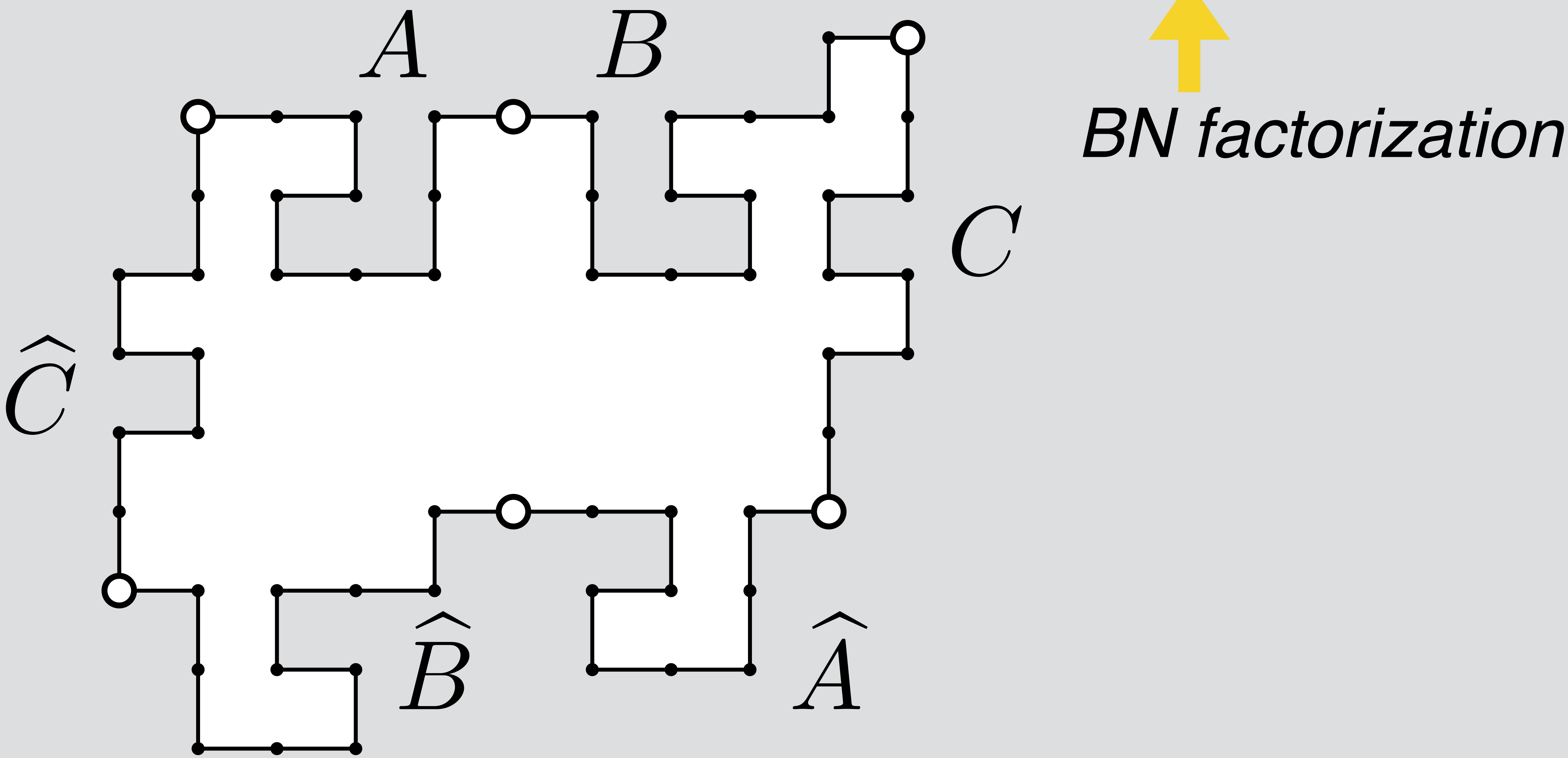

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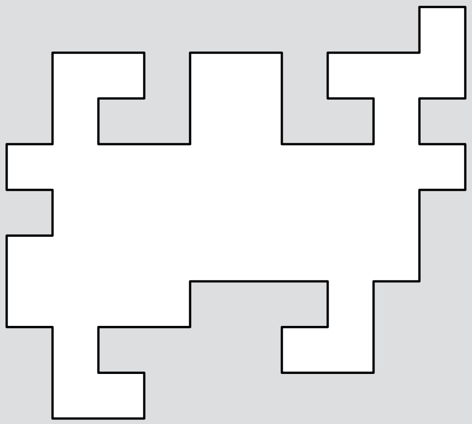
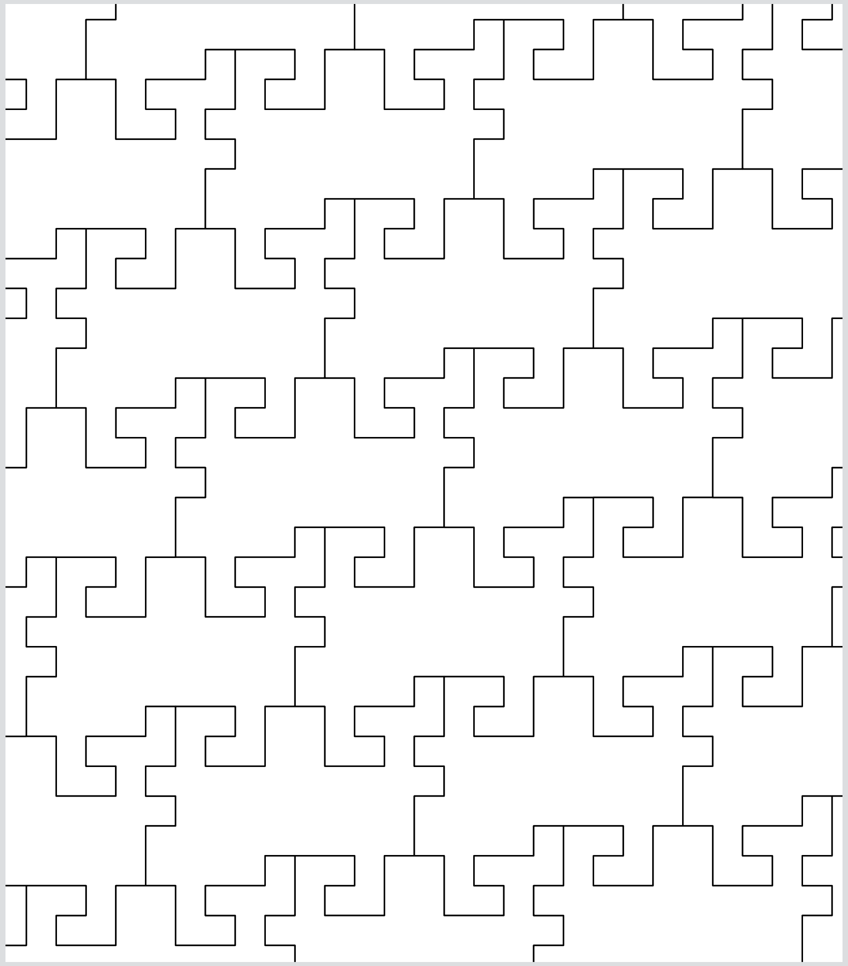
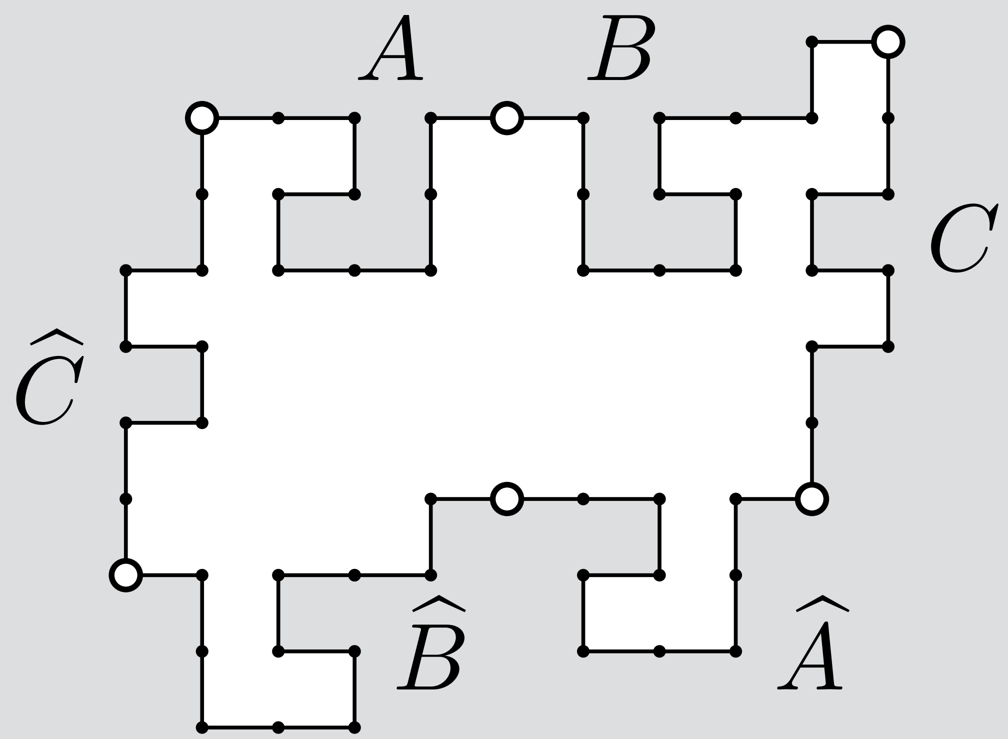


P has a plane tiling



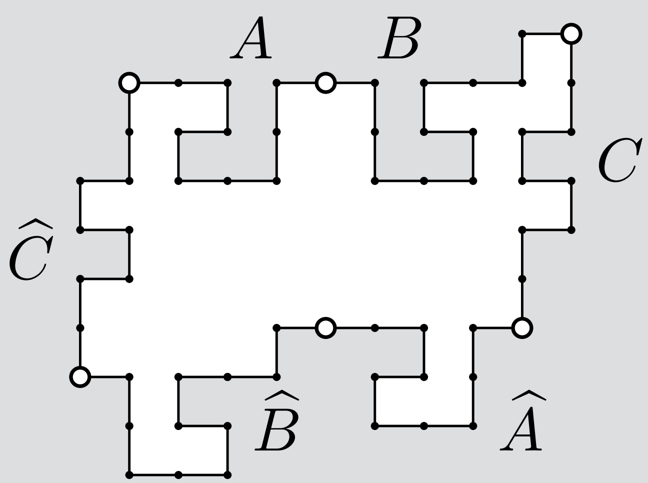
if and only if

Boundary word of P has BN factorization



Testing for BN factorization

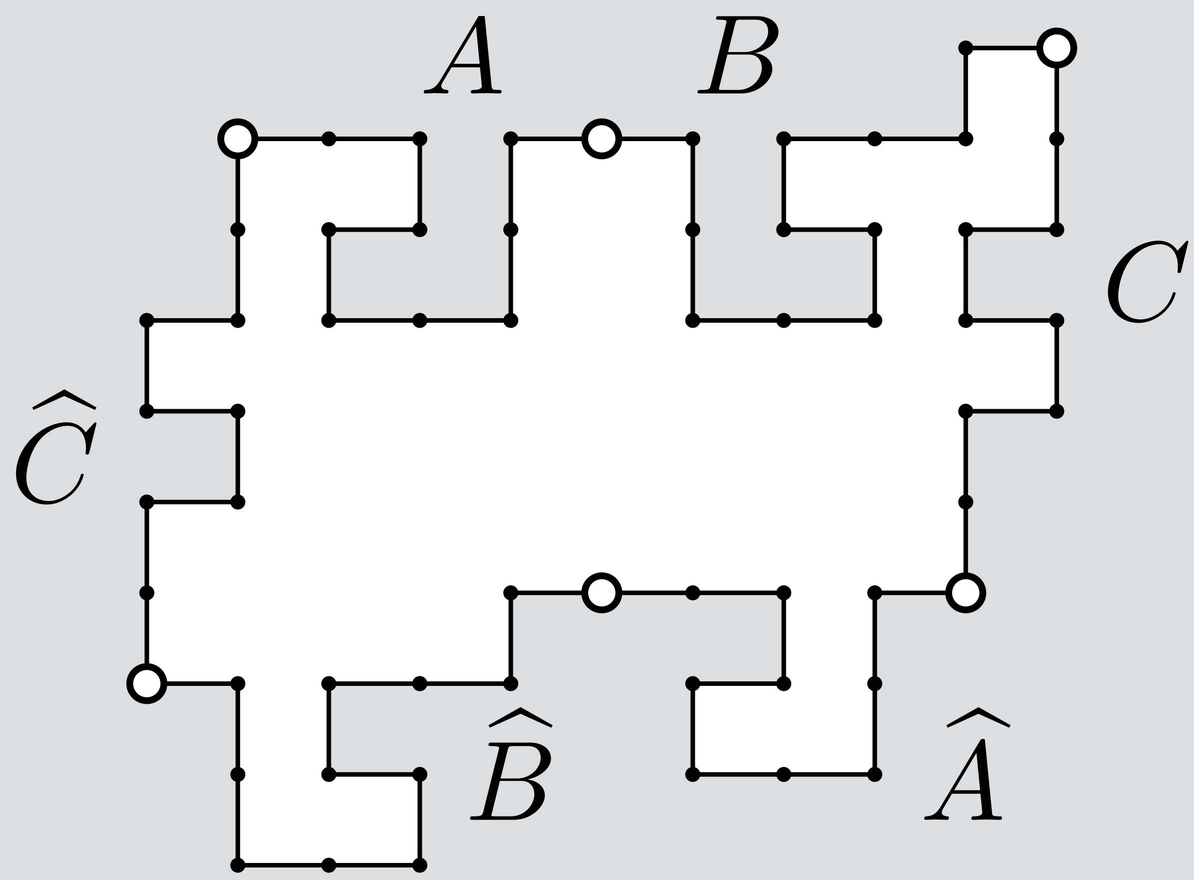
Given boundary word W with |W| = n, does W = $ABC\widehat{A}\widehat{B}\widehat{C}$?

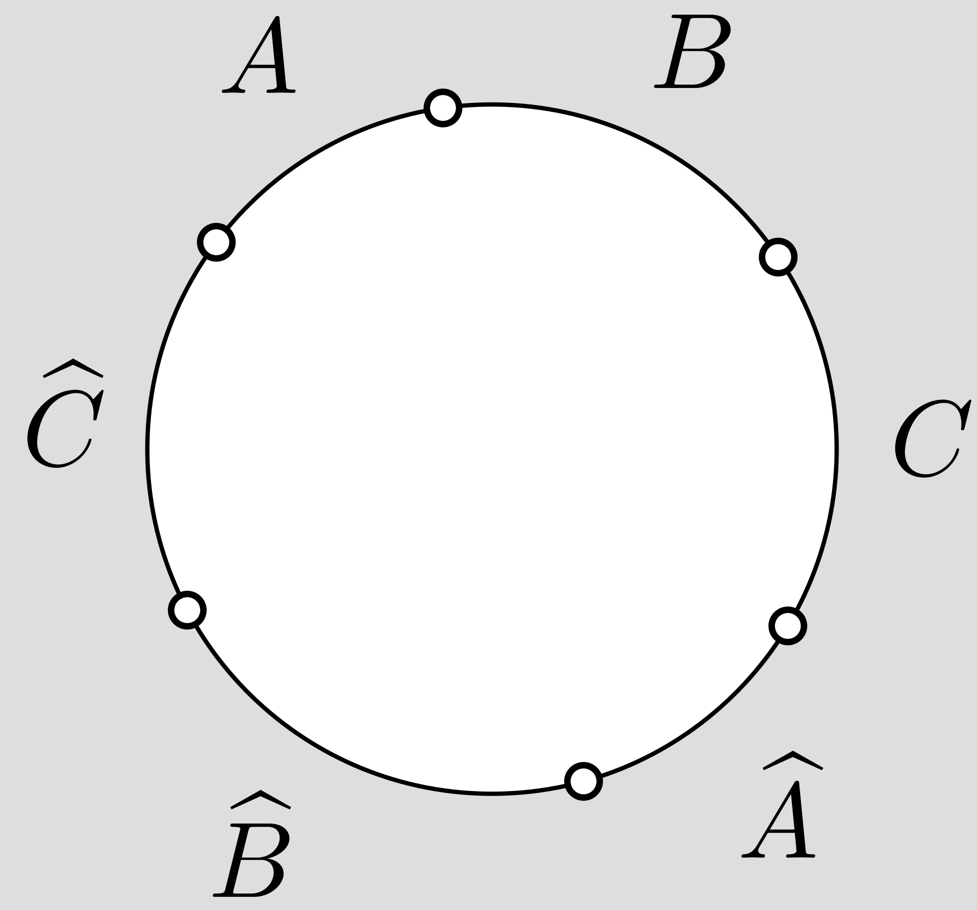


- [Gambini, Vuillon 2007]: $O(n^2)$
- [Provençal 2008]: $O(n*\log^3(n))$
- [Brlek, Provençal, Fédou 2009]: O(n) in two special cases.

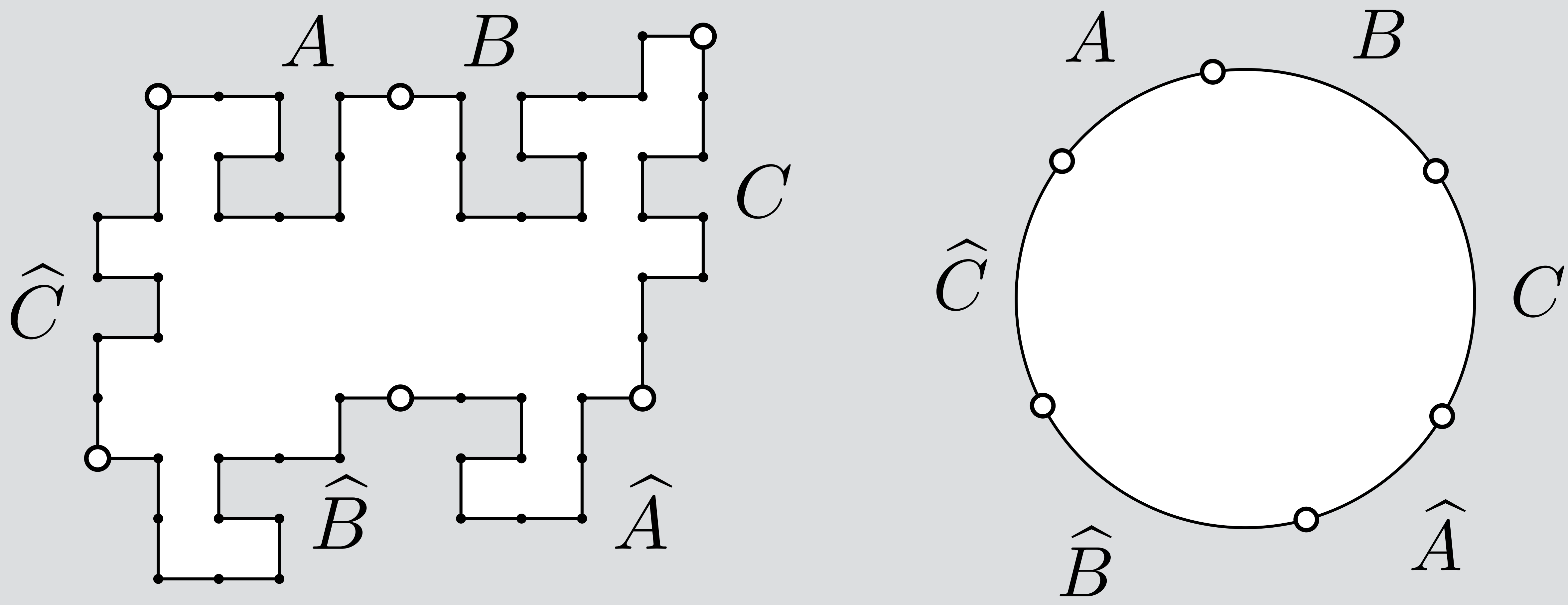
This work: O(n) algorithm for all inputs.

The algorithm



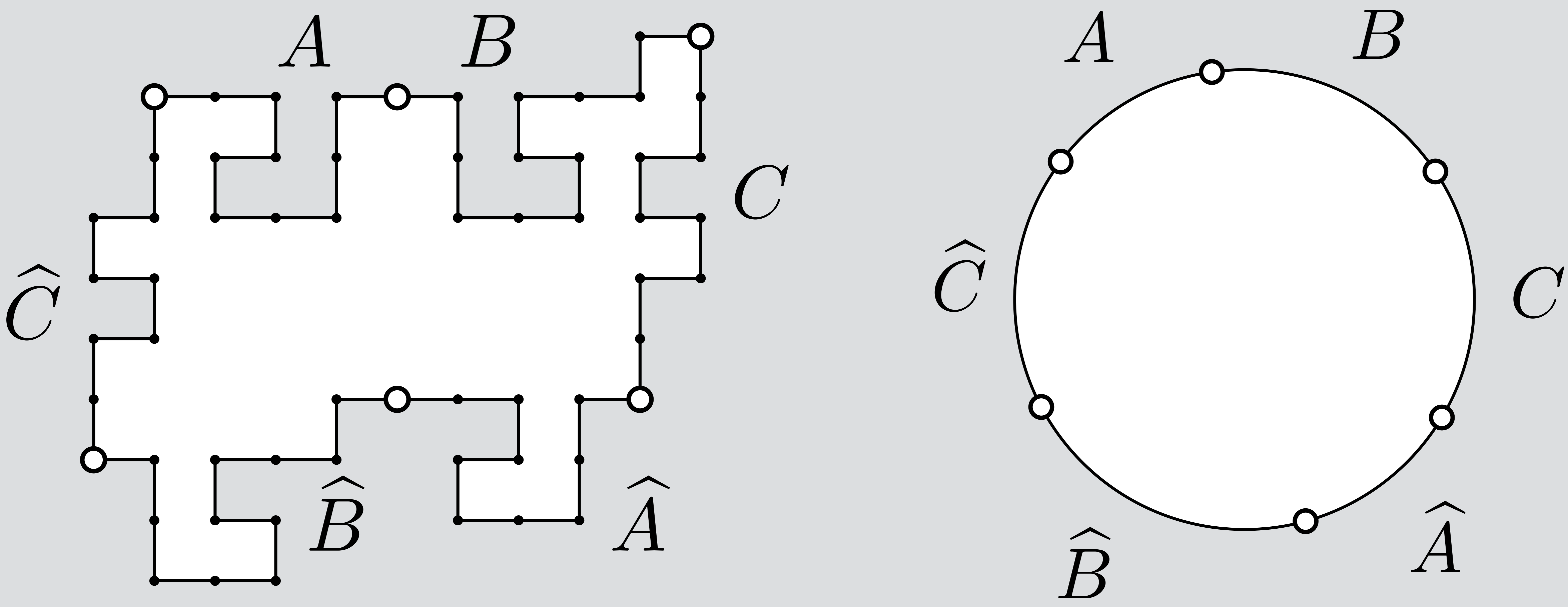


Factors come in pairs: $A, \widehat{A}$ & $B, \widehat{B}$ & $C, \widehat{C}$



Factors come in pairs: $A, \widehat{A}$ & $B, \widehat{B}$ & $C, \widehat{C}$

Each pair is centered around diametral locations.



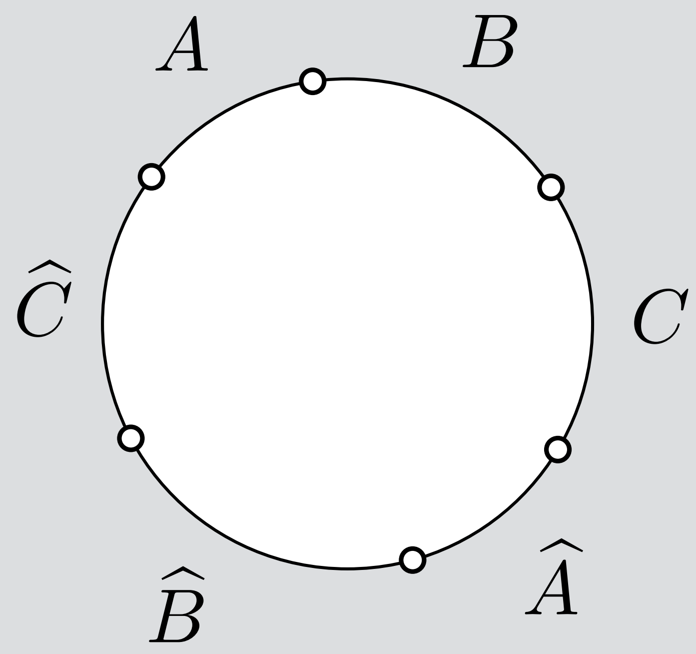
Factors come in pairs: $A, \widehat{A}$ & $B, \widehat{B}$ & $C, \widehat{C}$

Each pair is centered around diametral locations.

Pairs match as far along boundary as possible.

Admissible factors

<u>Admissible factor</u>: word A such that boundary word $W = AU\widehat{A}V$ with |U| = |V|, U[1] ≠ $\overline{\text{U[-1]}}$, V[1] ≠ $\overline{\text{V[-1]}}$.



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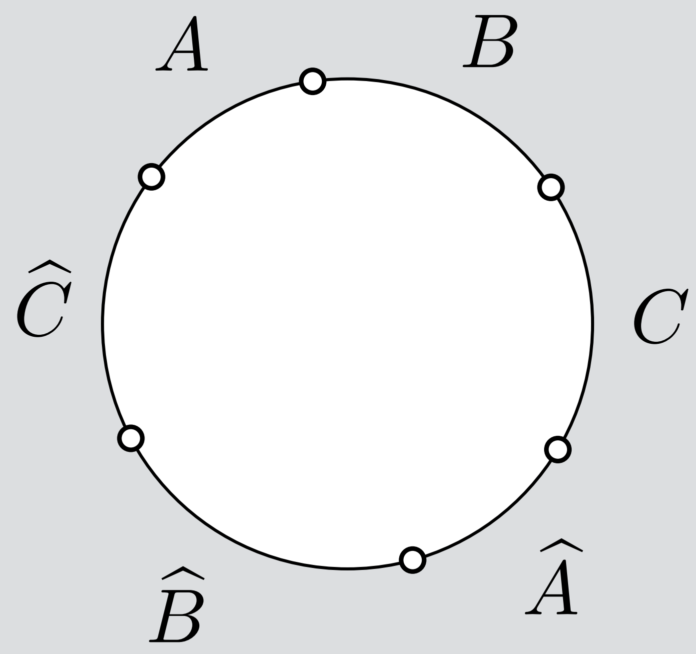
[Brlek et al. 2009]: Every factor of a BN factorization is admissible.



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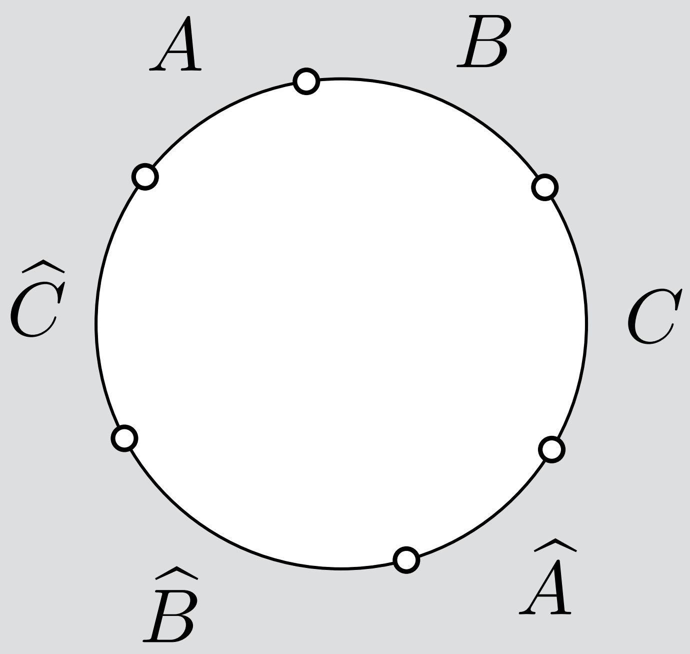

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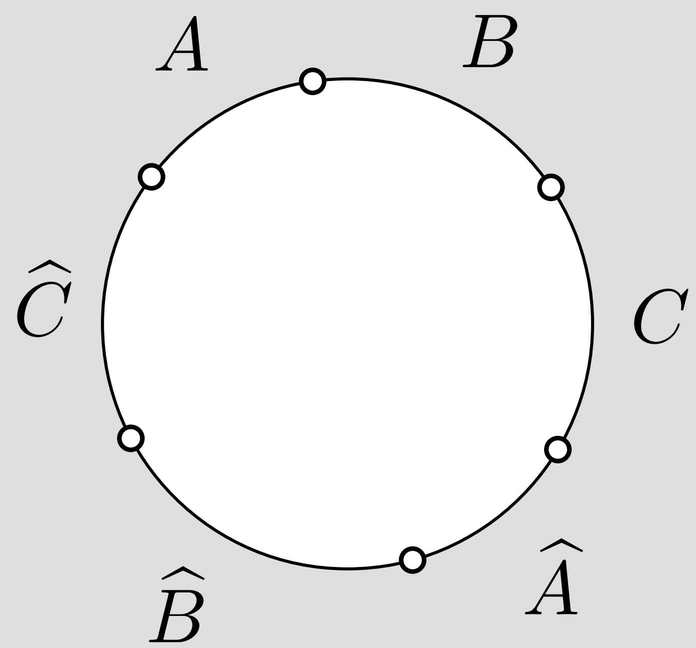

[Brlek et al. 2009]: can compute all O(n) admissible factors in O(n) time.

W has BN factorization $\Leftrightarrow$ W has consecutive admissible factors A, B, C with $|ABC| = |W|/2$.

The barrier

Given O(n) admissible factors of W, decide if W has consecutive admissible factors A, B, C with |ABC| = |W|/2.

The barrier

in O(n) time

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The barrier

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Given O(n) admissible factors of W, decide if W has consecutive admissible factors A, B, C with |ABC| = |W|/2.

The key idea comes from:

A Linear-Time On-Line Recognition Algorithm for "Palstar"

Journal of the Association for Computing Machinery, Vol 25, No 1, January 1978, pp 102–111

ZVI GALIL

Tel Aviv University, Tel Aviv, Israel

AND

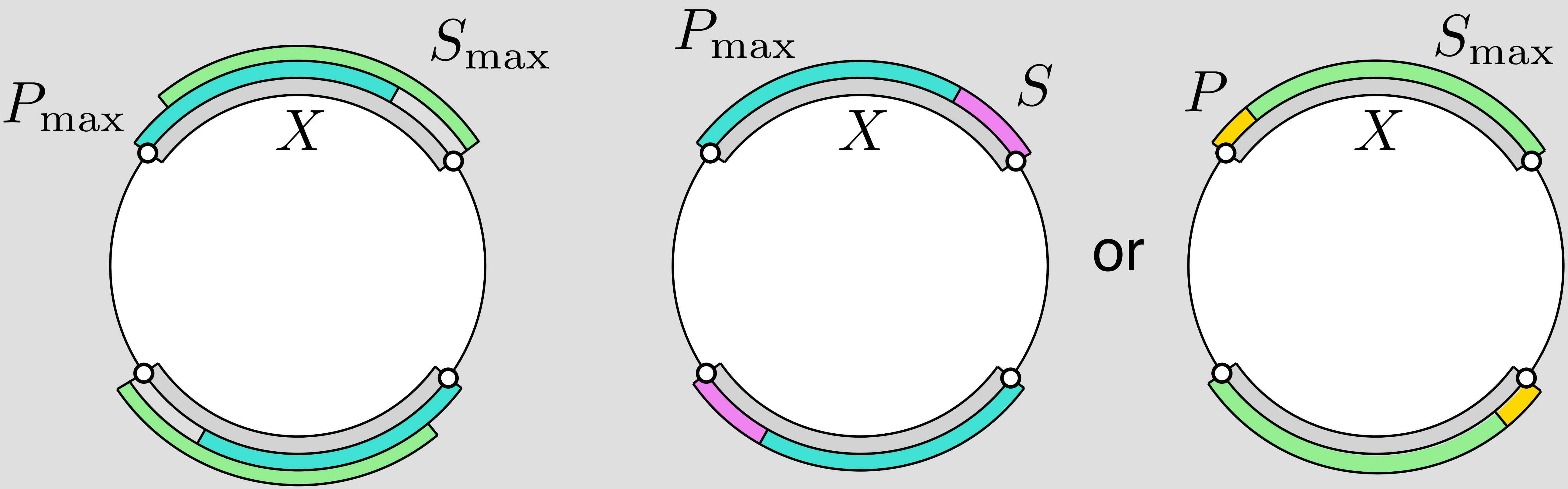
JOEL SEIFERAS

The Pennsylvania State University, University Park, Pennsylvania

Lemma: X = PS with P, S admissible ⇔

X = P_{max}S or X = PS_{max} with:

- P_{max} the longest prefix admissible factor of X, or
- S_{max} the longest suffix admissible factor of X.

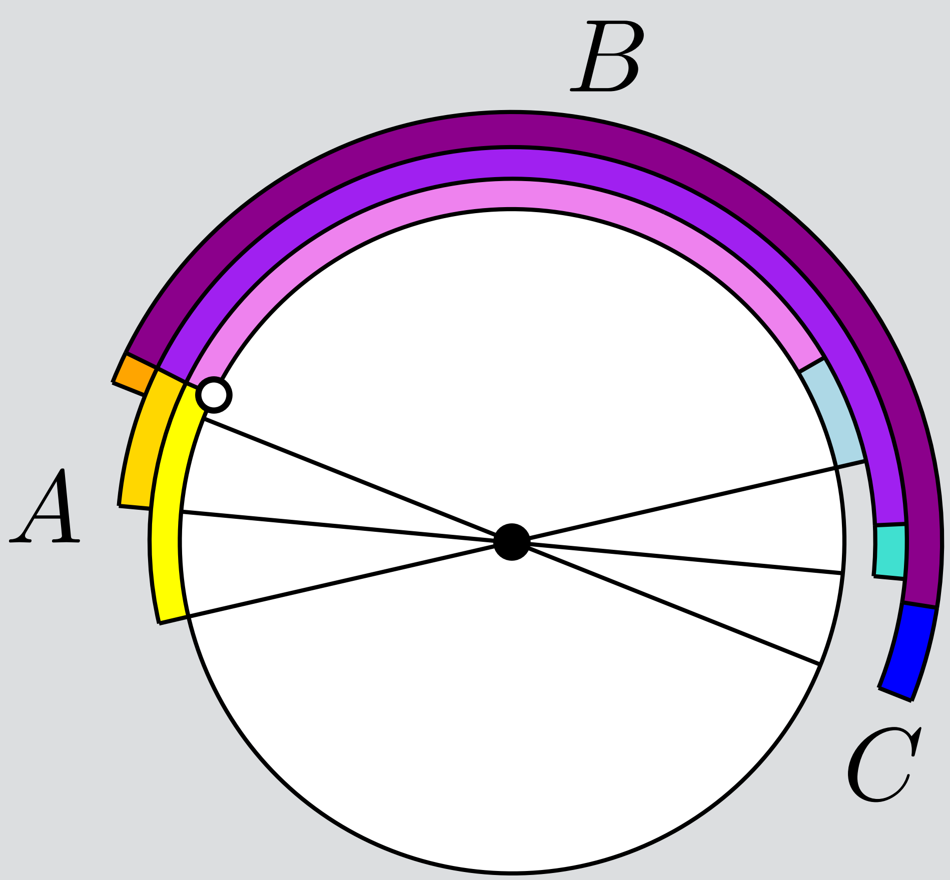


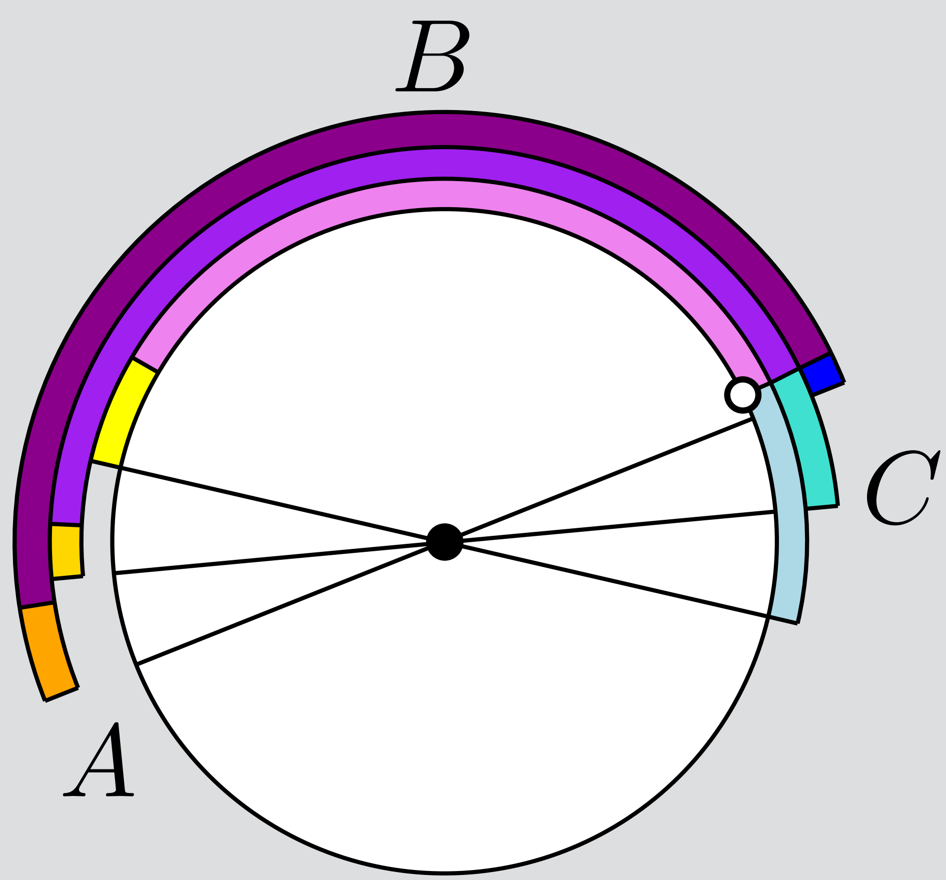
Proof nearly identical to similar result for palindromes by [Galil, Seiferas 1978]

Finding consecutive A,B,C with |ABC| = n/2.

1. For each A, search for longest B such that $|AB| \leq n/2$, check whether factor C with $|ABC| = n/2$ is admissible.

2. For each C, search for longest B such that $|BC| \leq n/2$, check whether factor A with $|ABC| = n/2$ is admissible.





Finding consecutive A,B,C with |ABC| = n/2.

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O(n) time using two-finger scans.

Algorithm

1. Compute all admissible factors.

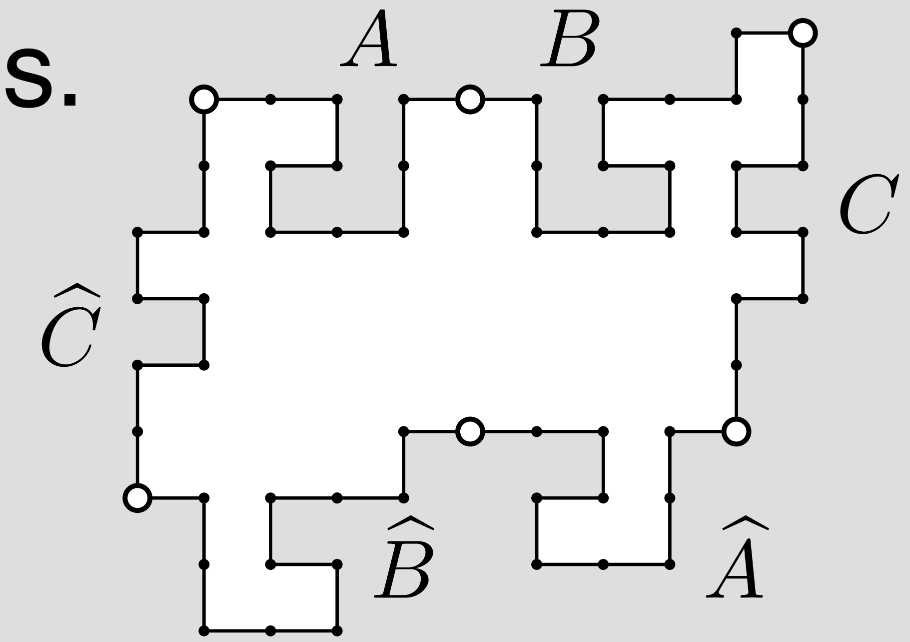
2. Sort admissible factors starting at each letter.
   Repeat for factors ending at each letter.

3. Two-finger scans to search for consecutive admissible factors A,B,C with $|ABC| = n/2$.

O(n)-time algorithm

Enumeration

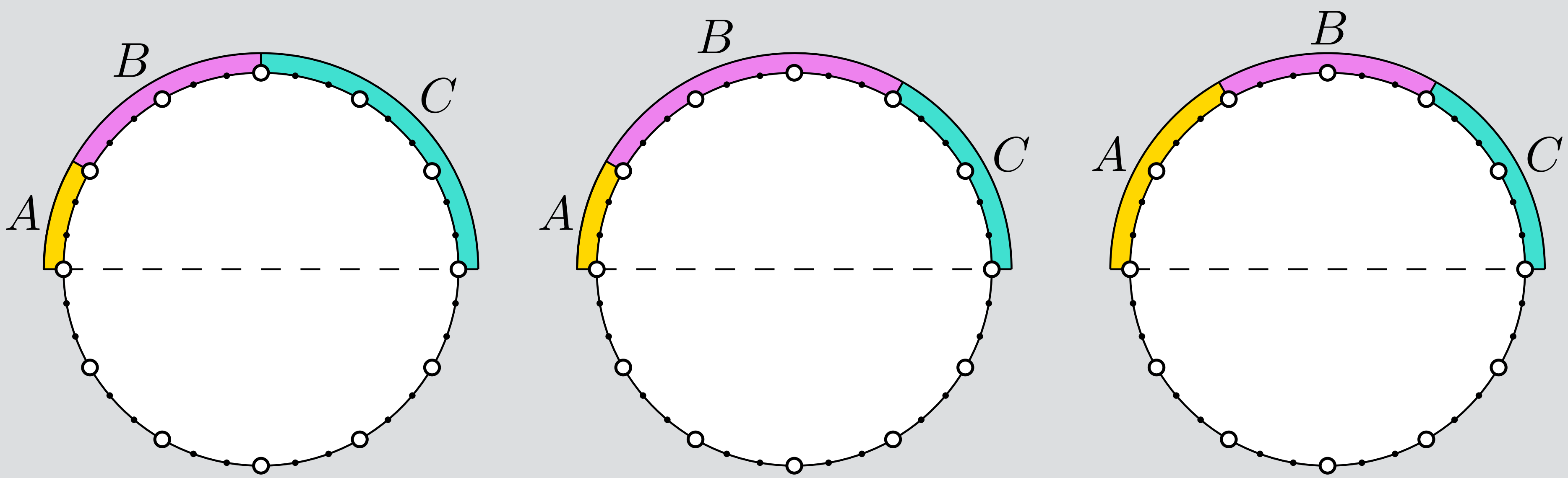
BN factorizations = isohedral tilings.



Algorithm can also enumerate
all k factorizations in O(n+k) time.

Is k = O(n)? Claimed by [Provençal 2008].
Proved here.

Without additional structure on admissible factors, $\Omega(n^{3/2})$ factorizations possible:



$\Theta(n^{1/2})$ locations, every pair defines an admissible factor

Pick diametral locations for start of A, end of C: $\Theta(n^{1/2})$

Pick start & end locations of B: $\Theta(n)$

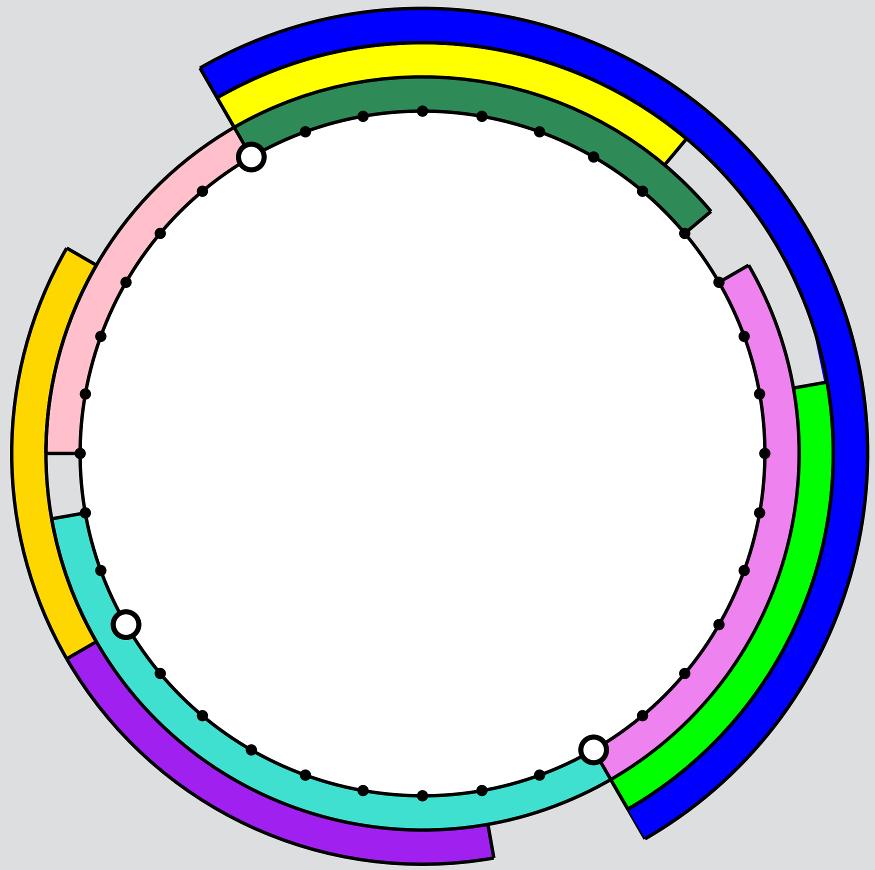
Need structure!

O(n) Factorizations

Lemma: there exists a set of O(1) locations in W such that every admissible factor with length ≥ |W|/6 either starts or ends at a location in the set.

By symmetry, O(1) locations where ABC starts.

O(n) choices of B per ABC ⇨ O(n) factorizations.



Conclusion

Optimal O(n) algorithm to decide if a polyomino can tile plane by translation (at all!)

- Key: 1978 result on palindromes

Algorithm also enumerates all such tilings that are isohedral.

Upcoming work: other factorization forms, extending to polygons.

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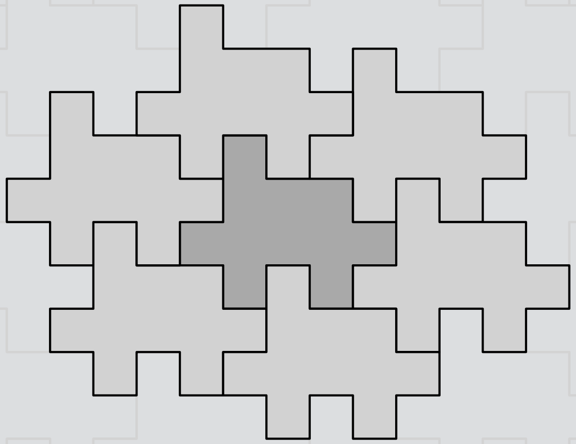

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