

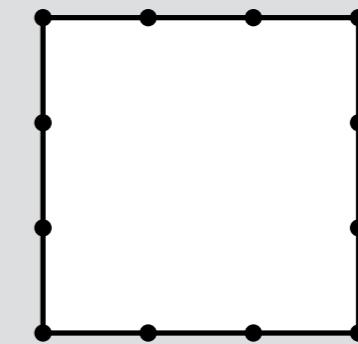
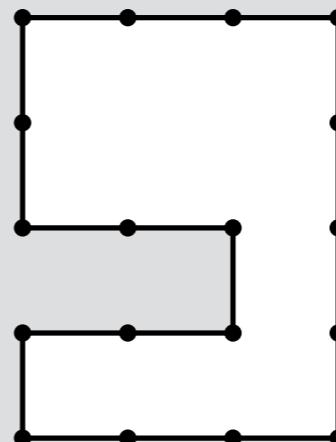
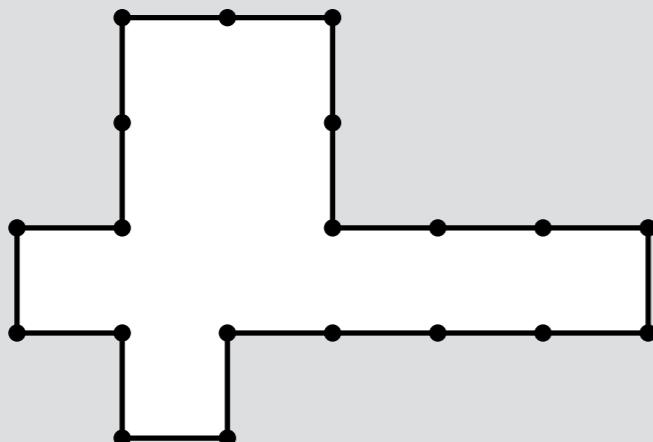
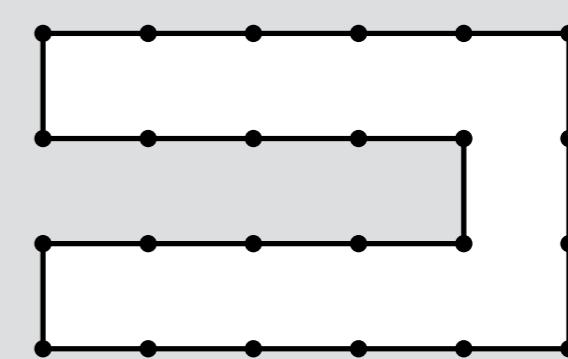
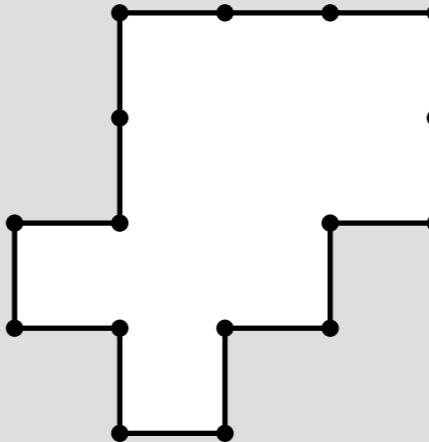
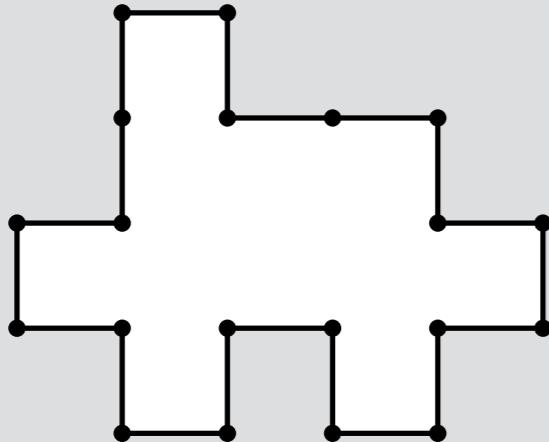
# A Quasilinear-Time Algorithm for Tiling the Plane Isohedrally with a Polyomino

Stefan Langerman<sup>\*1</sup> and Andrew Winslow<sup>1</sup>

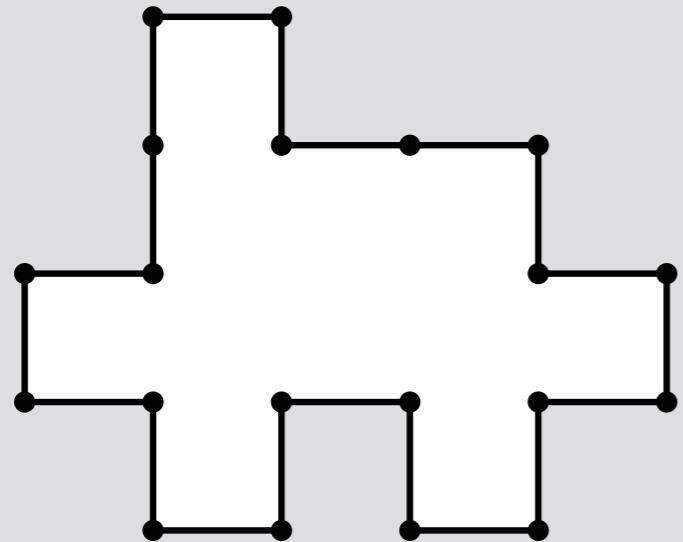
1 Département d’Informatique, Université Libre de Bruxelles,  
ULB CP212, boulevard du Triomphe, 1050 Bruxelles, Belgium,  
`{stefan.langerman, andrew.winslow}@ulb.ac.be`

# Polyominoes

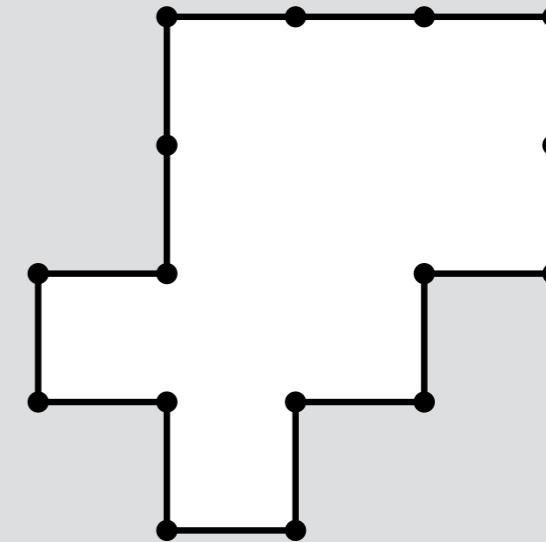
Rectilinear simple polygons with unit edge lengths



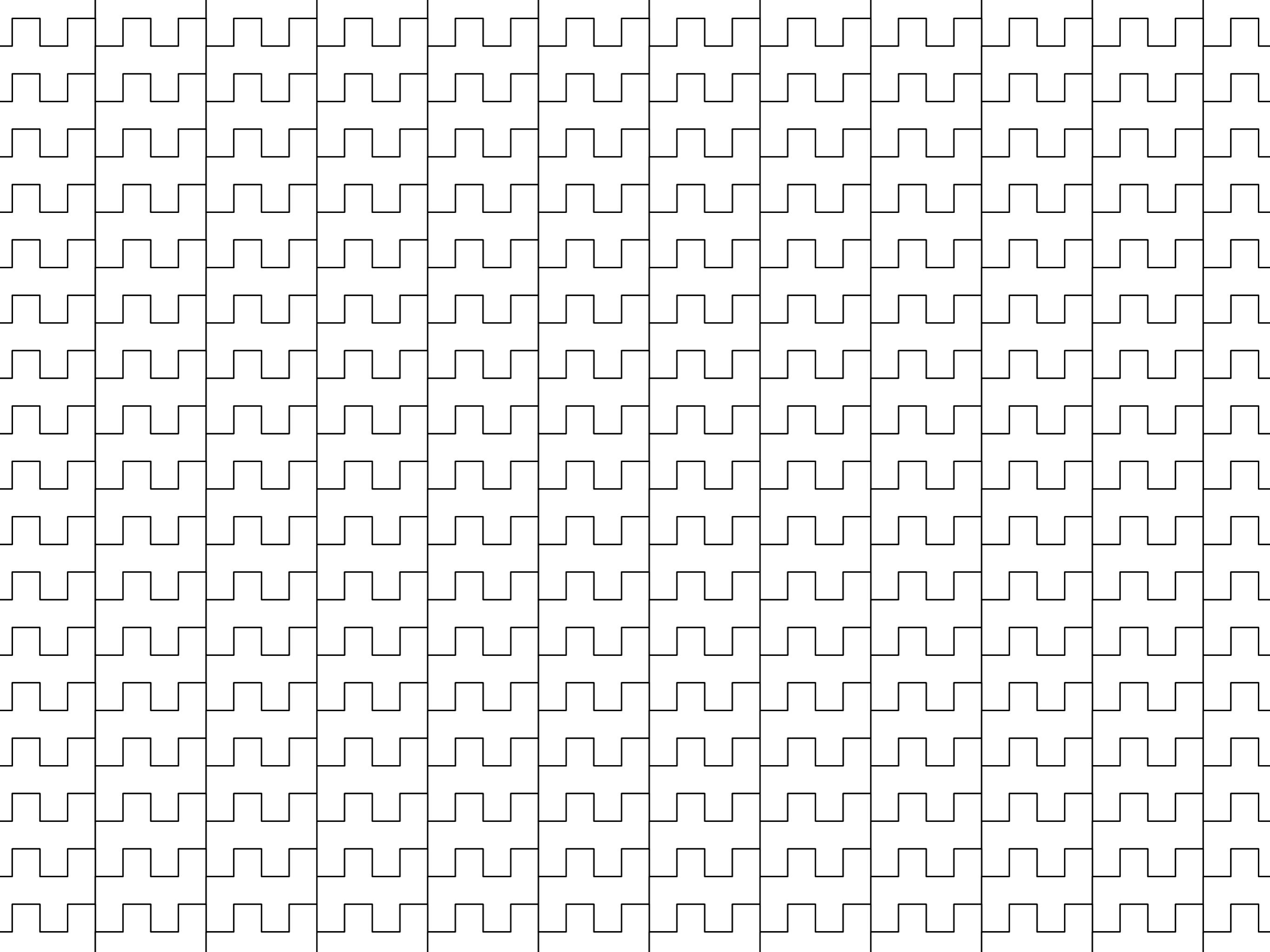
# Boundary words

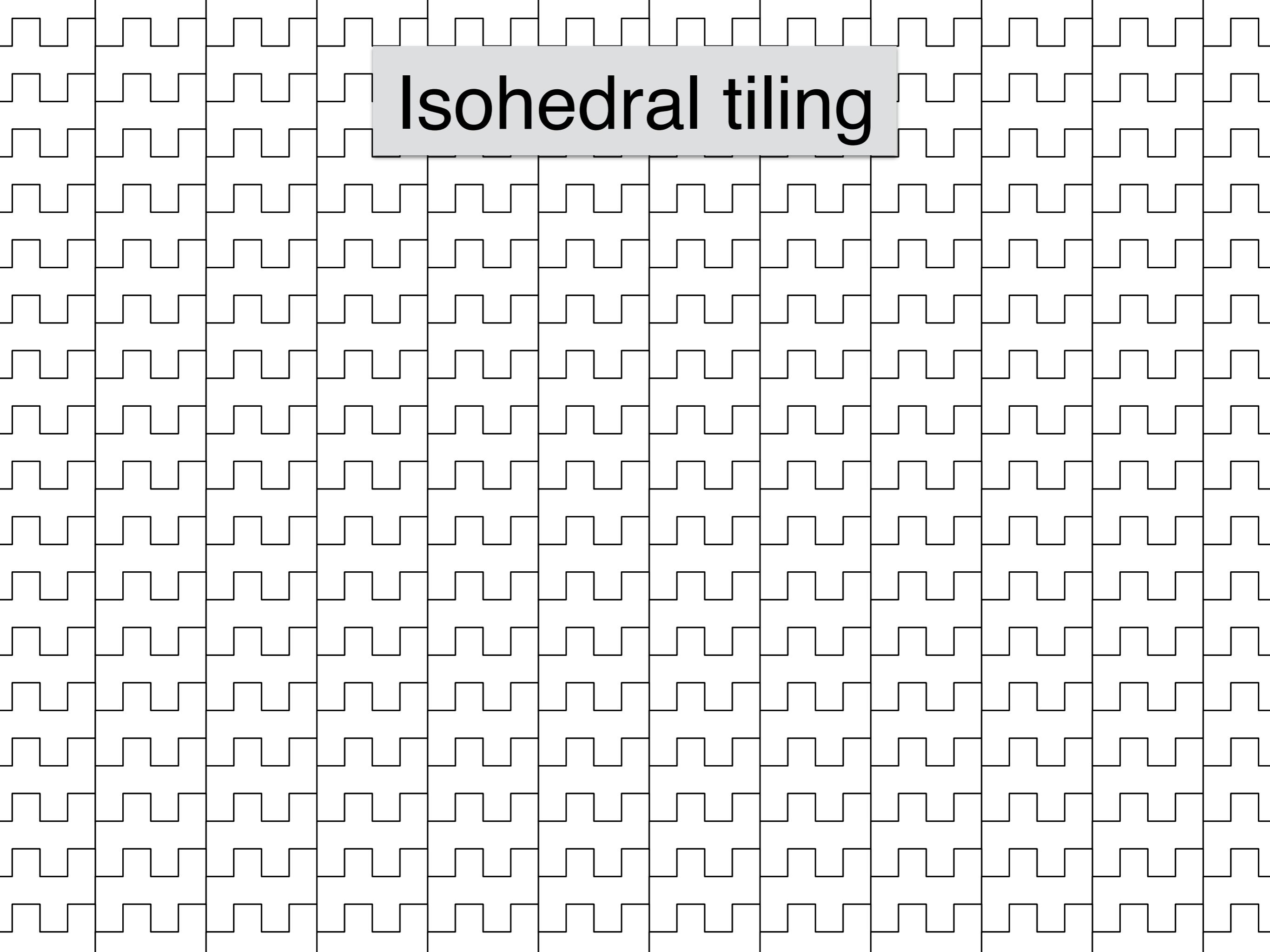


$$uru^2rdr^2drd(ldlu)^2l$$



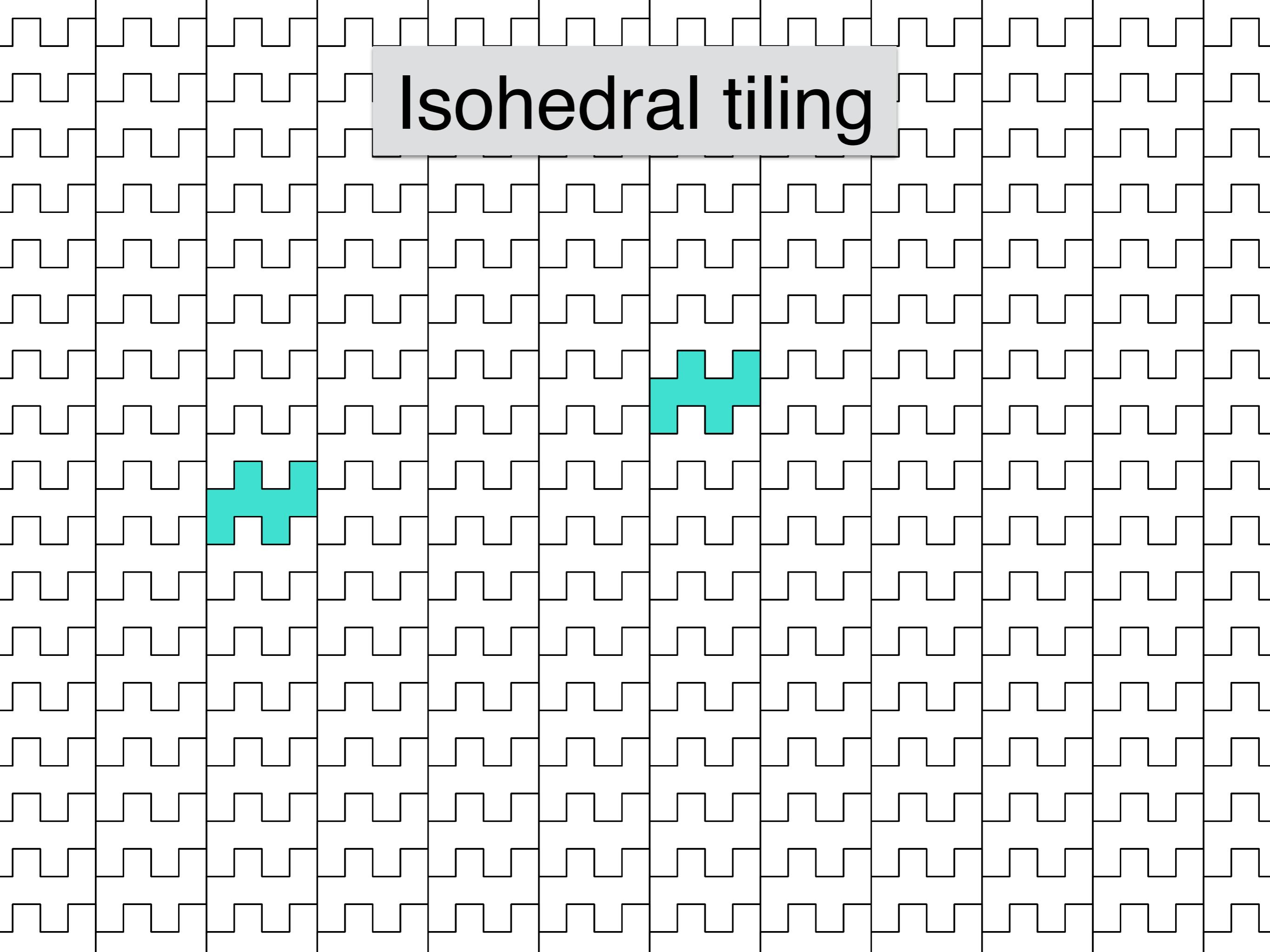
$$d(dl)^3 uluru^2 r^3$$



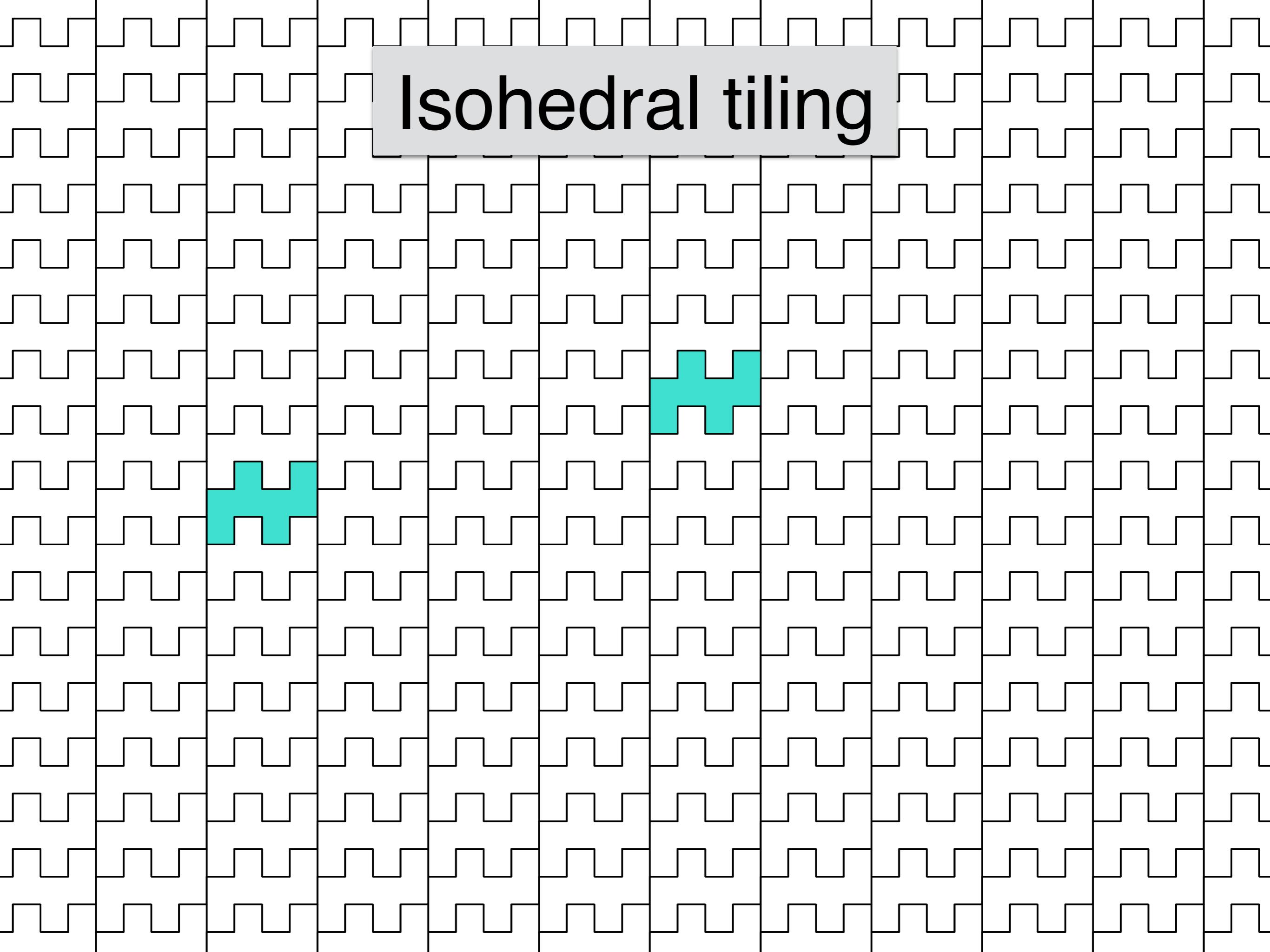


# Isohedral tiling

# Isohedral tiling



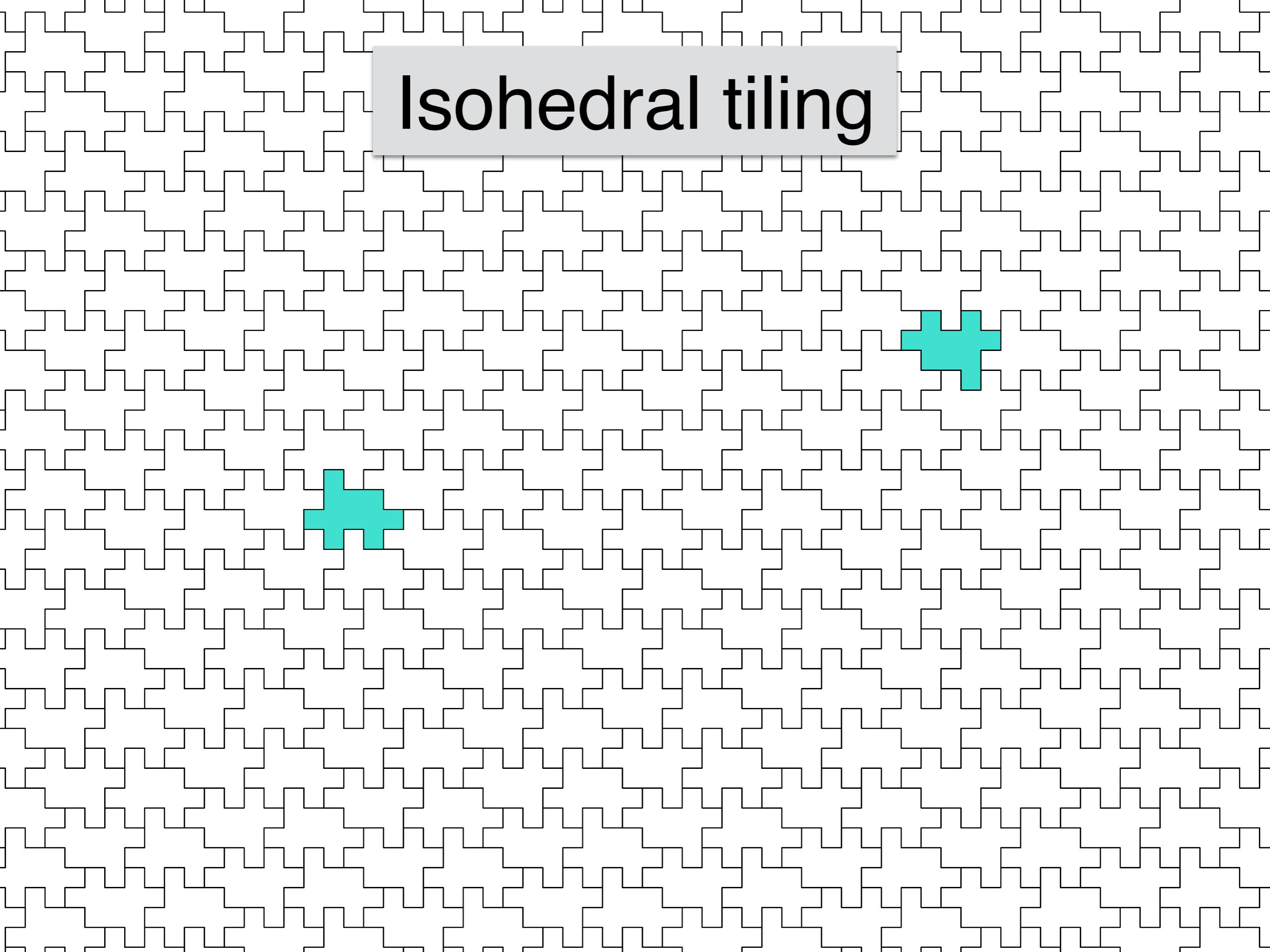
# Isohedral tiling



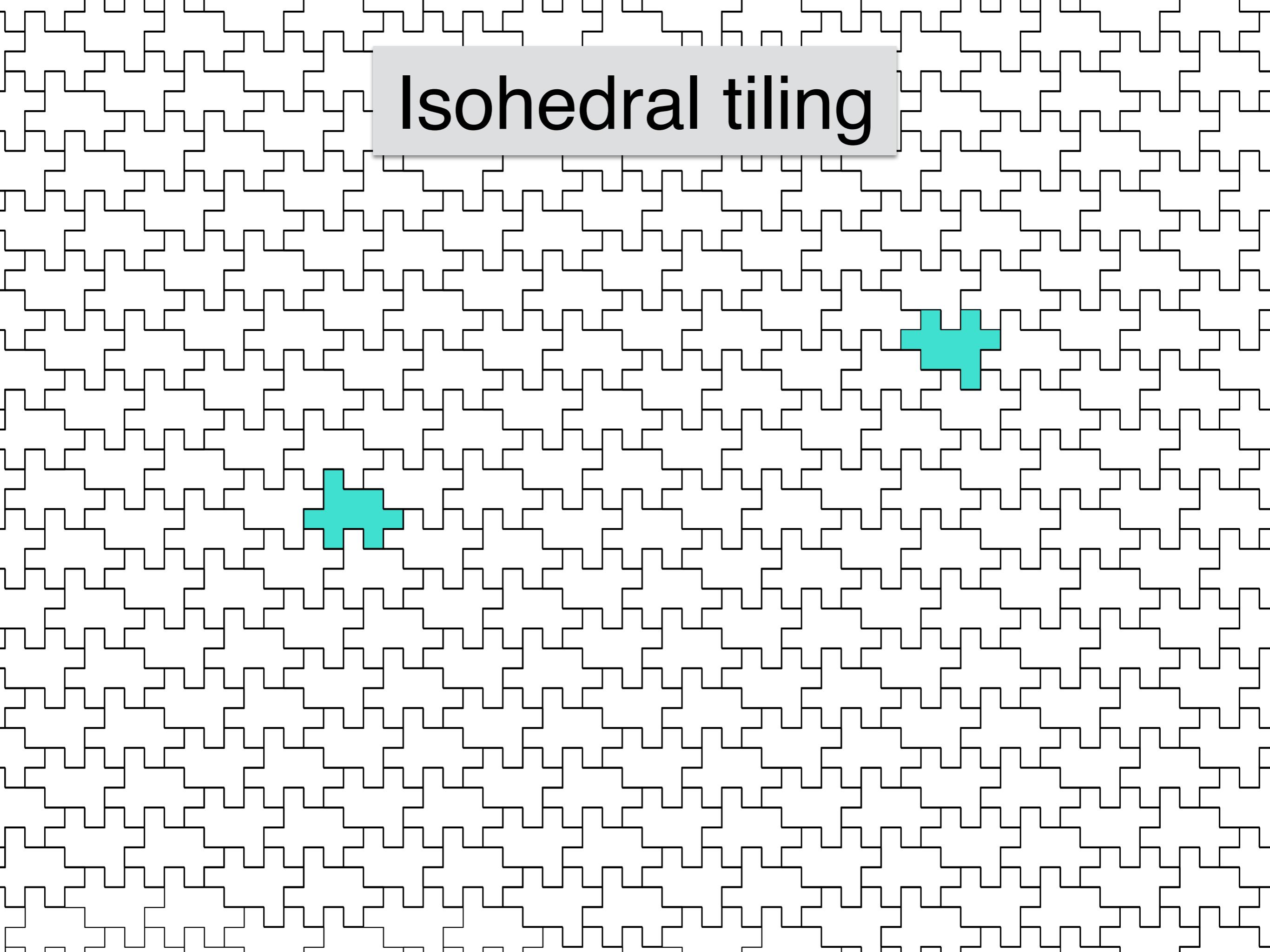
# Isohedral tiling



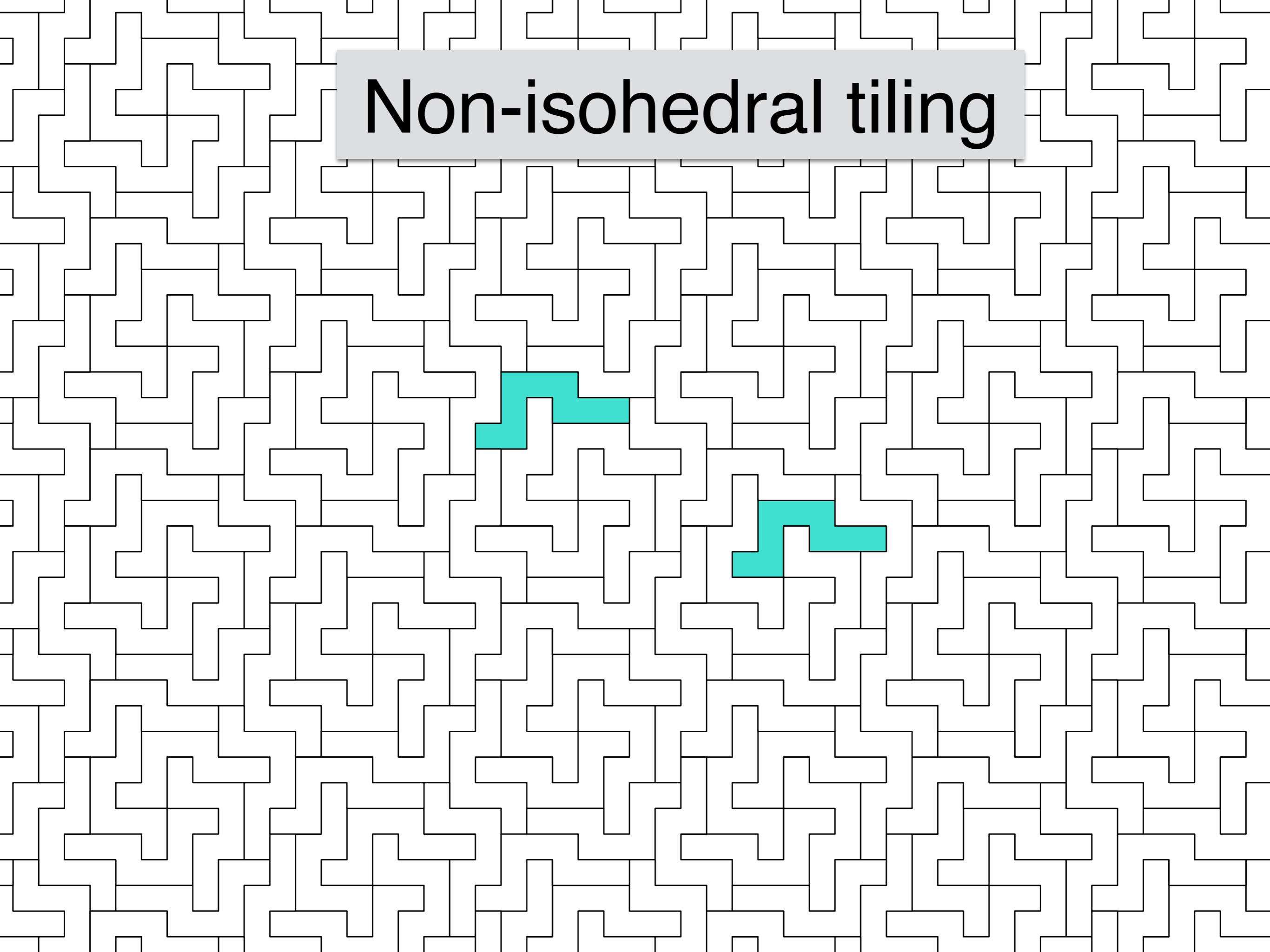
# Isohedral tiling



# Isohedral tiling

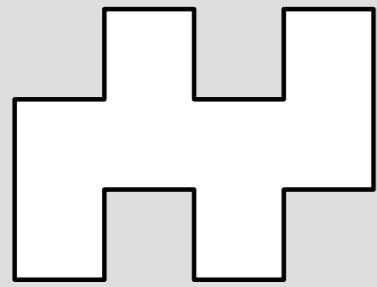


# Non-isohedral tiling

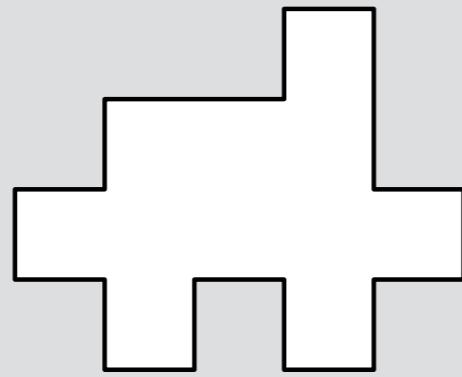


# Problem

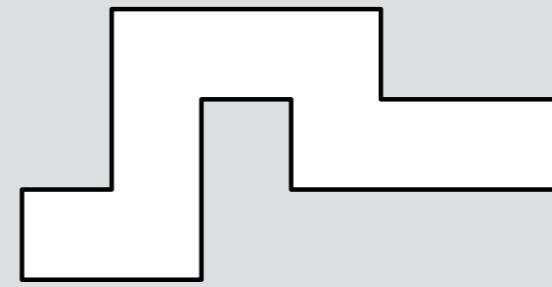
Decide whether a polyomino  
has an isohedral tiling.



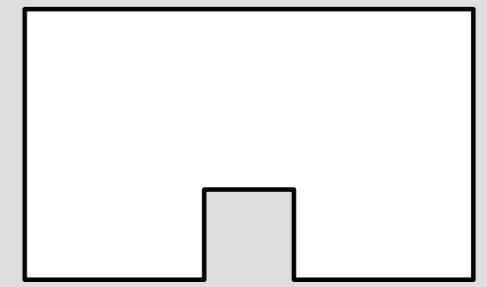
Yes



Yes

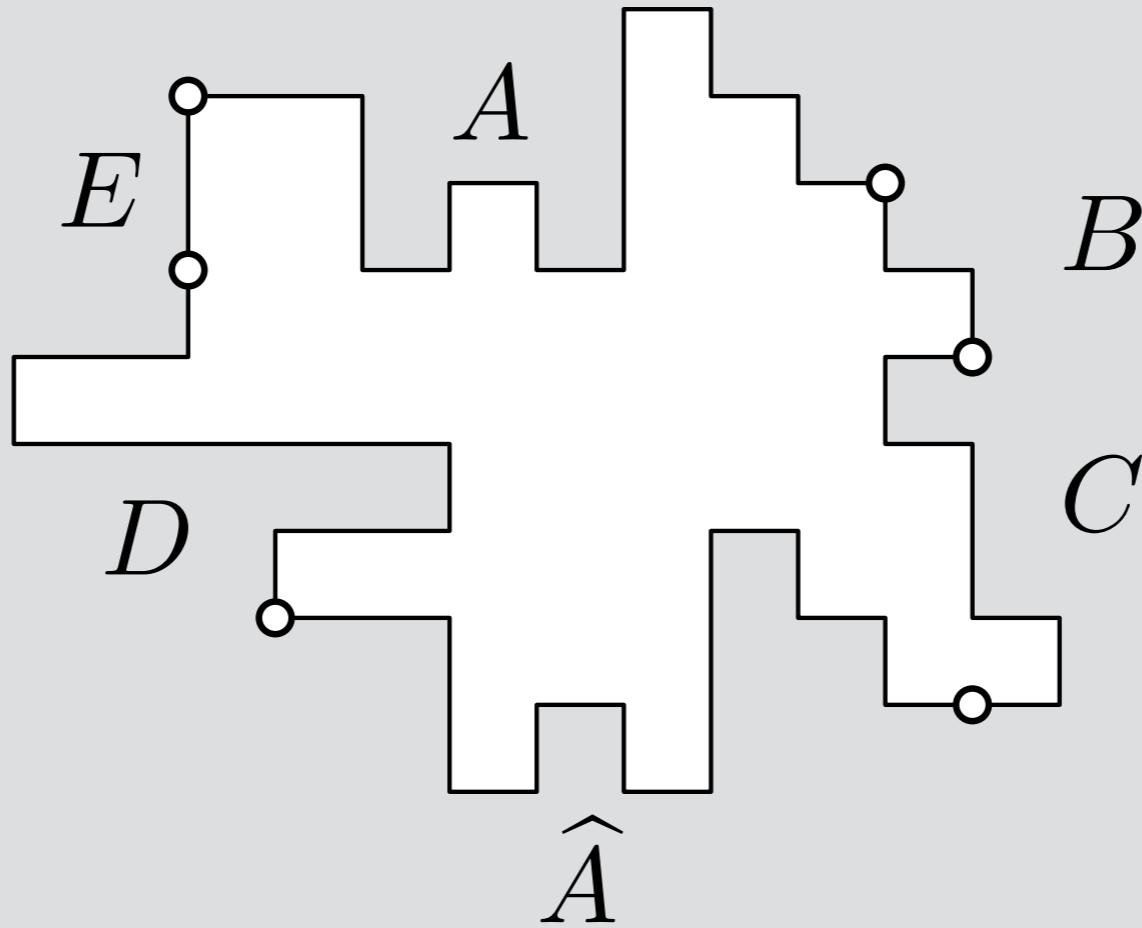


No



No

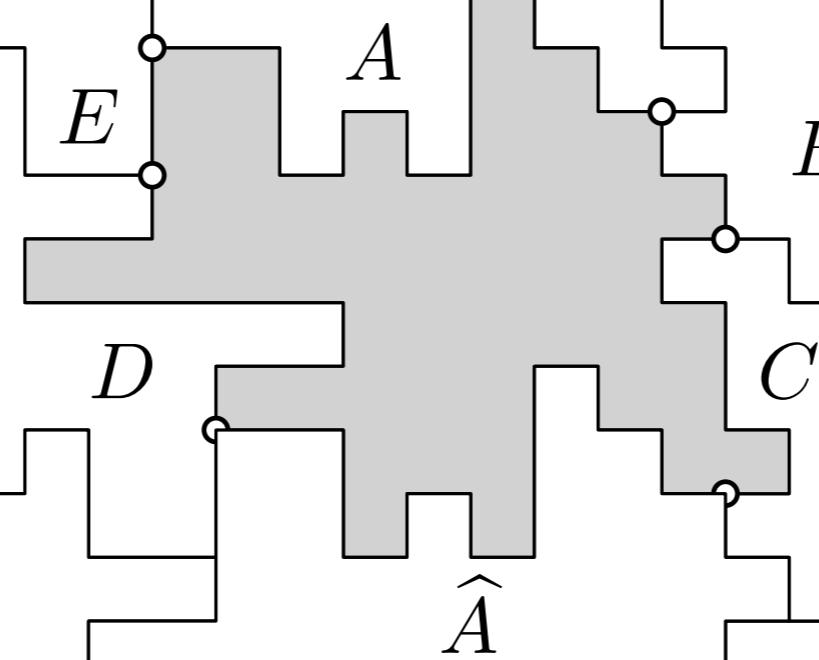
# Conway's criterion



$B, C, D, E$  palindromes

$$X = x_1 x_2 \dots x_n \quad \text{with} \quad \begin{array}{ll} \bar{u} = d & \bar{r} = l \\ \bar{X} = \bar{x}_n \bar{x}_{n-1} \dots \bar{x}_1 & \bar{d} = u \quad \bar{l} = r \end{array}$$

# Conway's criterion



# Isohedral boundary criteria

Tafel 10. Die 28 Grundtypen des Flächenschlusses										
Netzecken	6	5	4	3						
Netze	333333	63333	43433	44333	6363	6434	4444	666	884	12, 12, 3
p1										
p2										
p3										
p6										
p4										
gr										
pg										
pgg										

Die starke Umrandung umfaßt die 3 Haupttypen, von denen die anderen durch Schrumpfung von Linien oder Linienpaaren entstanden gedacht werden können.

Die Nummer rechts unten in jedem Feld ist die Nummer des zugehörigen Einzelbrates, S. 6 bis 77.

Netzecke      → Drehpunkt einer C-Linie

[Heesch, Kienzle 1963]

# Isohedral boundary criteria

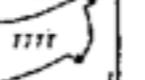
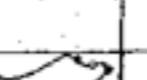
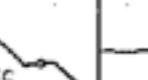
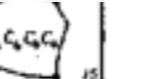
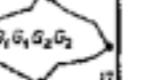
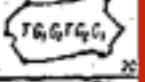
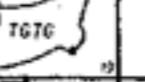
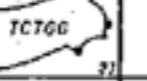
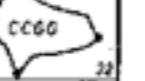
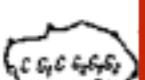
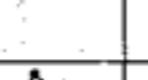
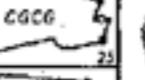
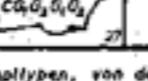
Die starke Umwandlung umfaßt die 3 Haupttypen, von denen die anderen durch Schrumpfung von Linien oder Liniengräben entstanden gedacht werden können.

Die Nummer rechts unten in jedem Feld ist die Nummer der zu bearbeitenden Formblätter 5-54-22

—+—- Hitzzecke —o—- Ohrhantl einer C-Linie

# [Heesch, Kienzle 1963]

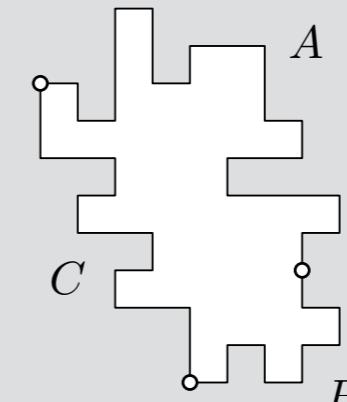
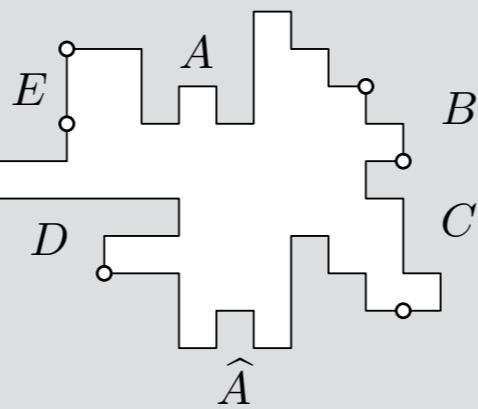
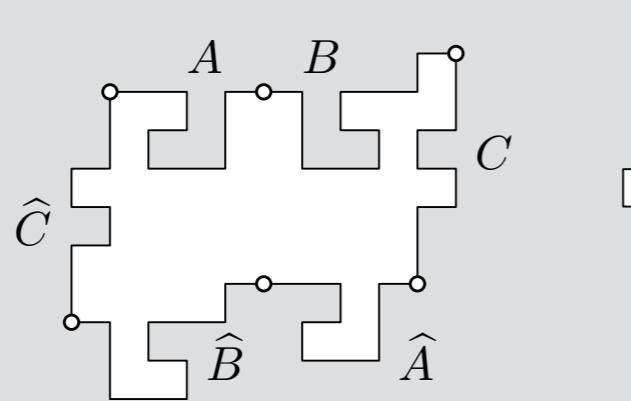
# Isohedral boundary criteria

Tafel 10. Die 28 Grundtypen des Flächenschlusses										
Netzecken	6	5	4	3						
Netze	333333	63333	43433	44333	6363	6434	4444	666	884	12, 12, 3
p1										
p2										
p3										
p6										
p4										
pg										
pg										
pgg										
pgg										
										

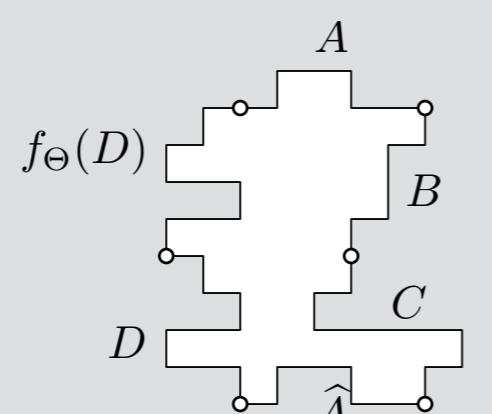
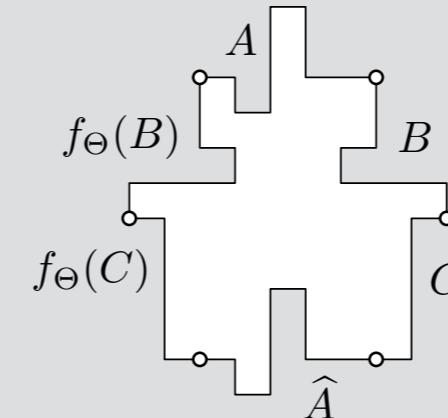
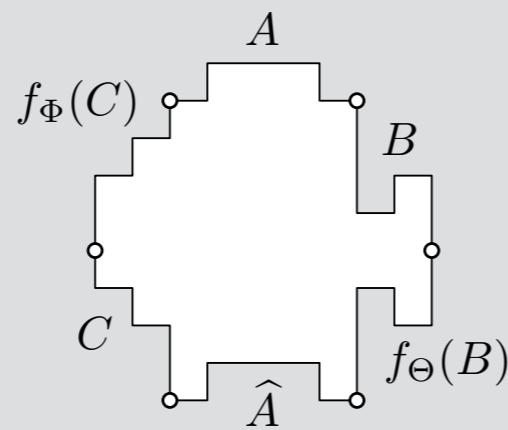
(not polyomino)

# [Heesch, Kienzle 1963]

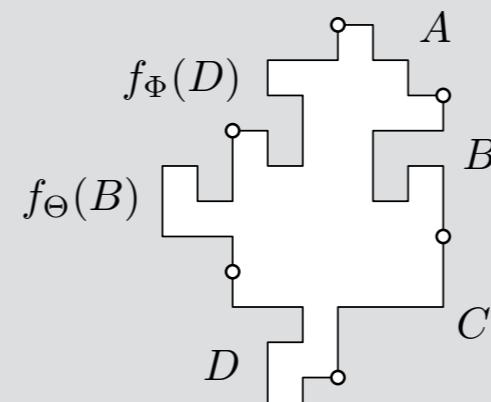
# 7 isohedral criteria



$B, C, D, E$  palindromes    $A, \hat{B}$  90-dromes,  $C$  palindrome

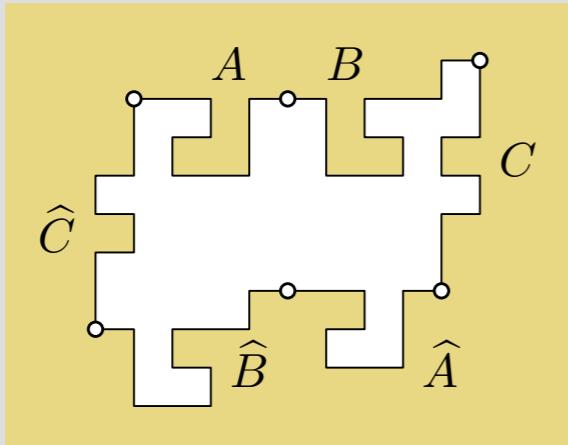


$B, C$  palindromes

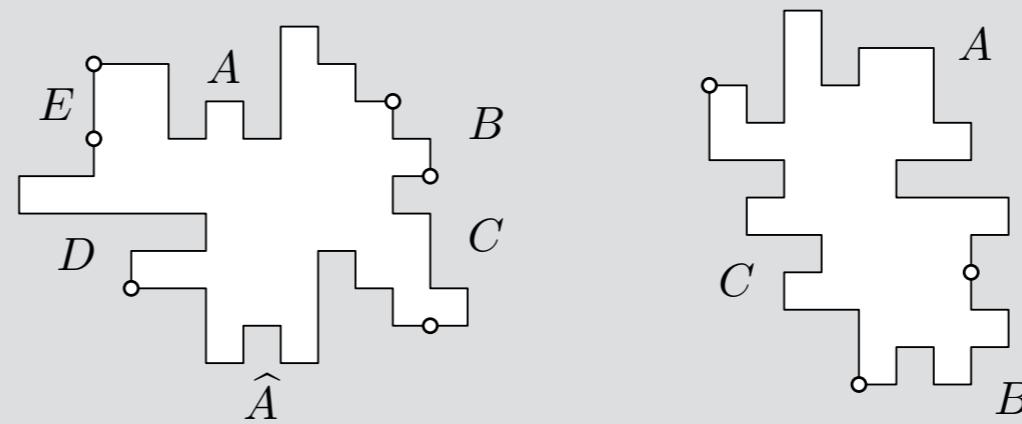


$A, C$  palindromes  
 $\Theta^\circ - \Phi^\circ = \pm 90^\circ$

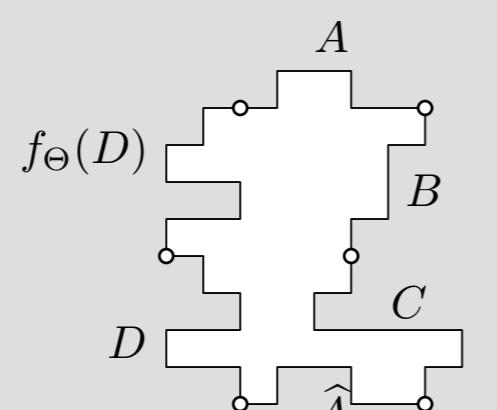
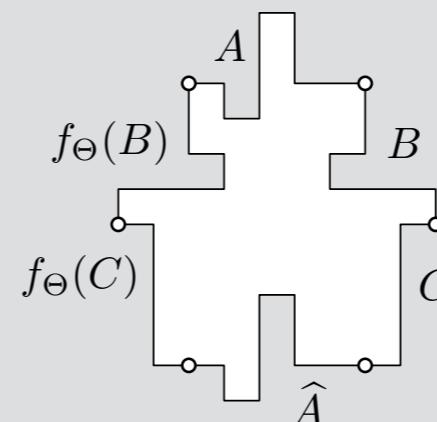
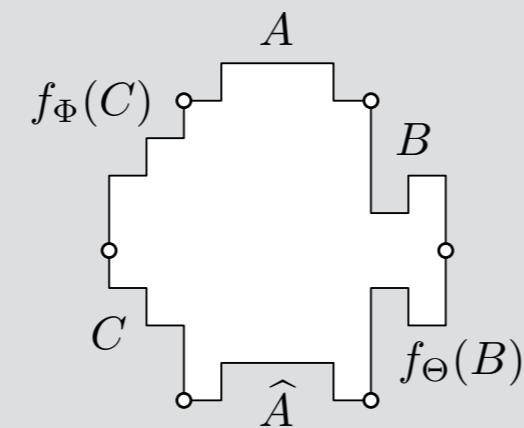
# 7 isohedral criteria



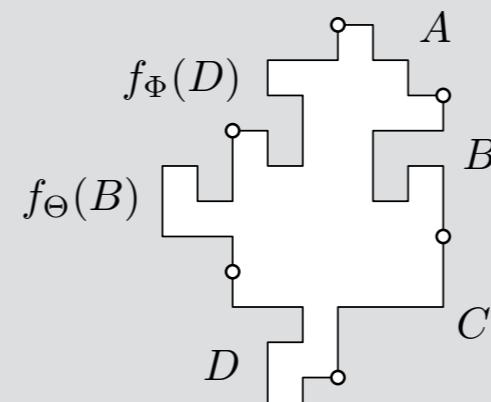
Translation criterion



$B, C, D, E$  palindromes    $A, B$  90-dromes,  $C$  palindrome

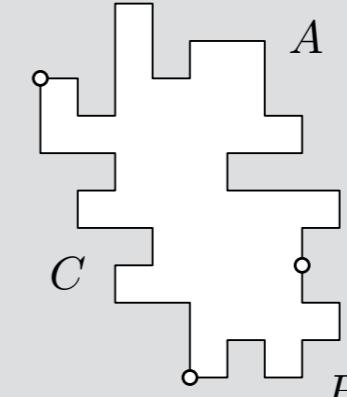
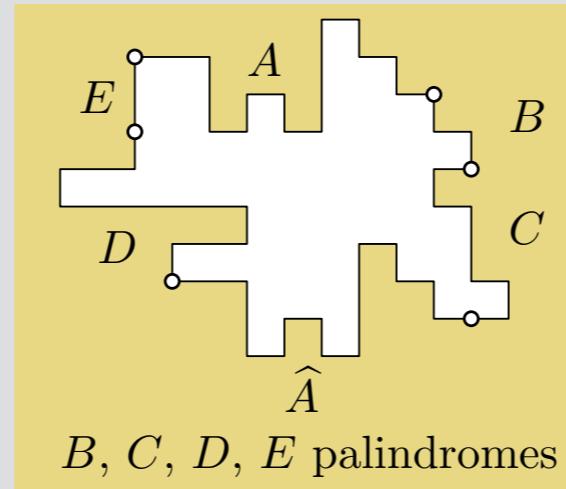
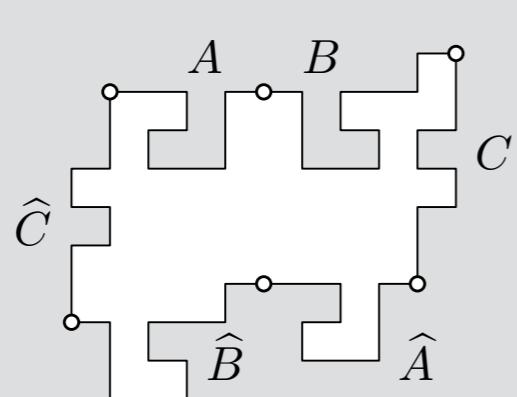


$B, C$  palindromes

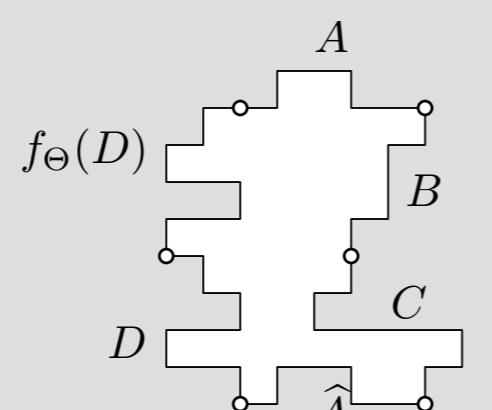
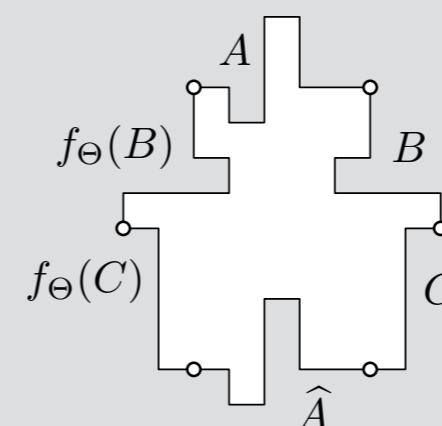
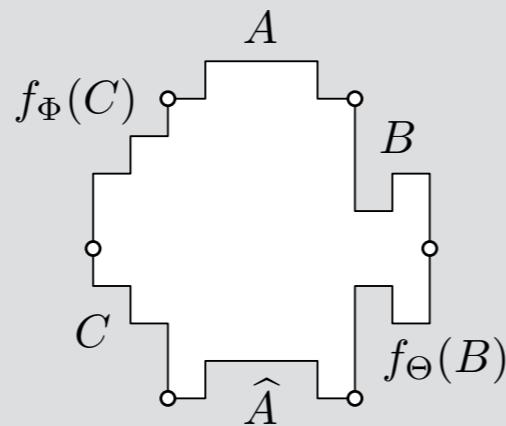


$A, C$  palindromes  
 $\Theta^\circ - \Phi^\circ = \pm 90^\circ$

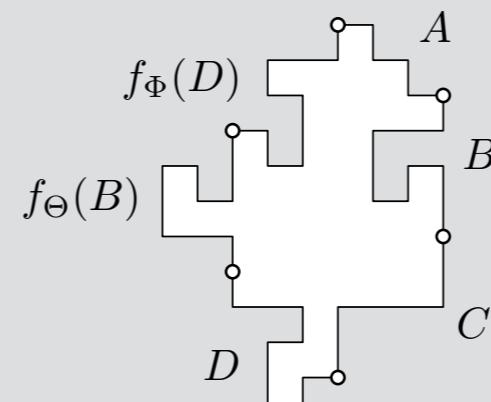
# 7 isohedral criteria



Conway's criterion



B, C palindromes

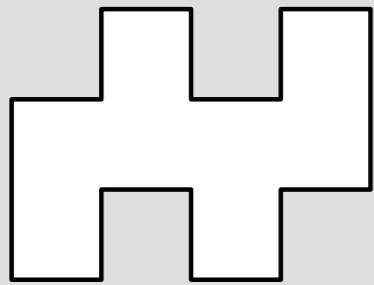


A, C palindromes  
 $\Theta^\circ - \Phi^\circ = \pm 90^\circ$

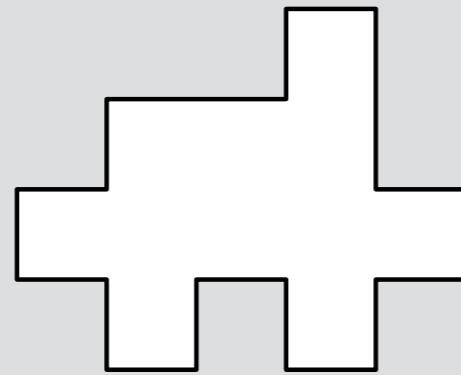
# Problem

Decide whether a polyomino  
has an isohedral tiling.

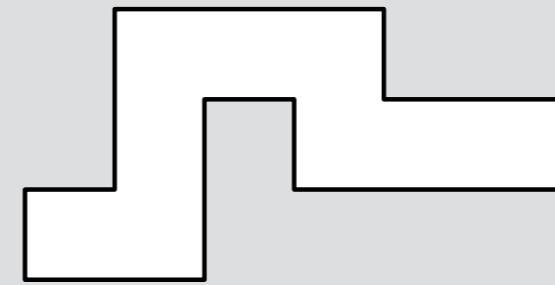
(passes any of 7 criteria)



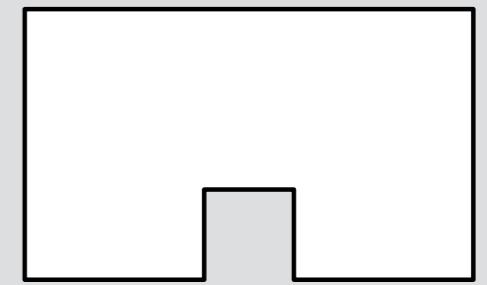
Yes



Yes



No



No

# Prior work

(input polyomino with n sides)

# Prior work

(input polyomino with n sides)

General case (all 7 criteria):

- [Keating, Vince 1999]:  $O(n^{18})$
- Naive checking of criteria:  $O(n^6)$
- This work:  $O(n * \log^2(n))$

# Prior work

(input polyomino with n sides)

General case (all 7 criteria):

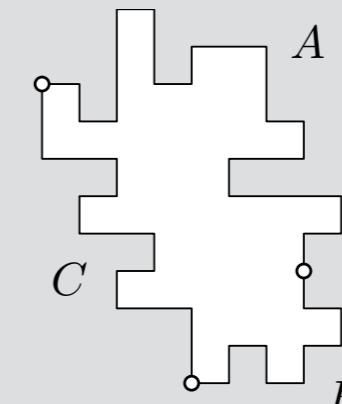
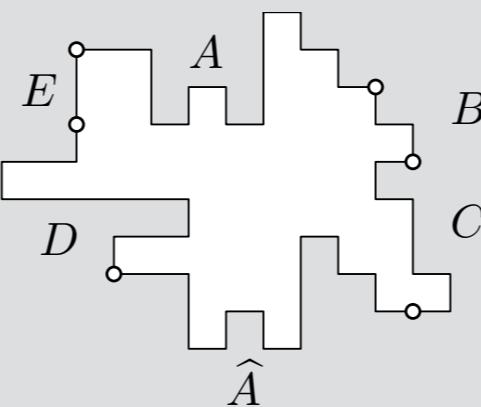
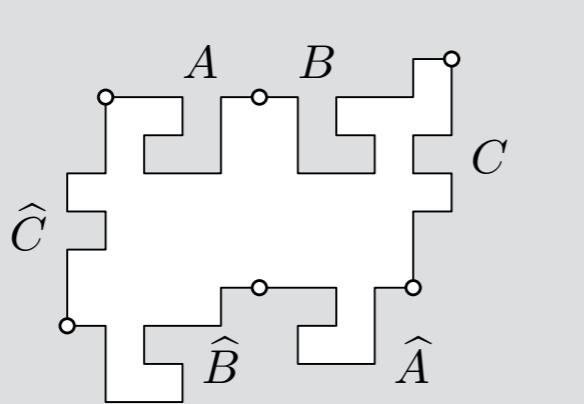
- [Keating, Vince 1999]:  $O(n^{18})$
- Naive checking of criteria:  $O(n^6)$
- This work:  $O(n * \log^2(n))$

Translation criterion only:

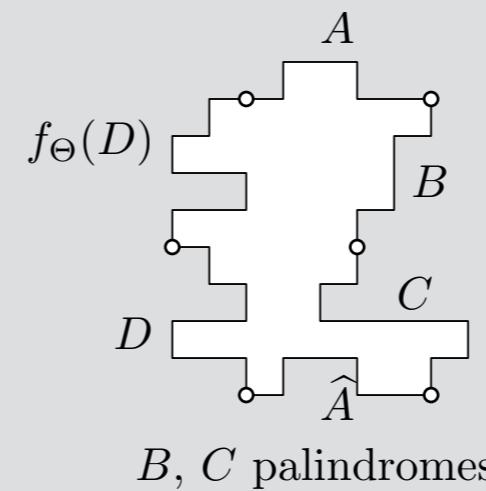
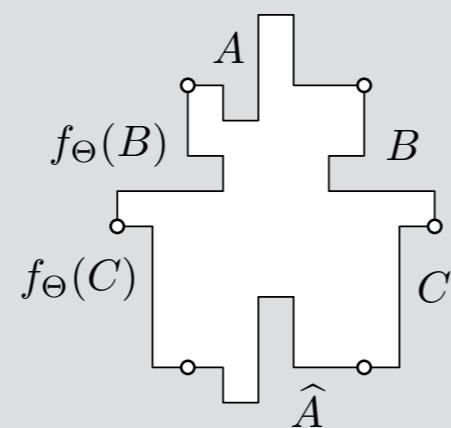
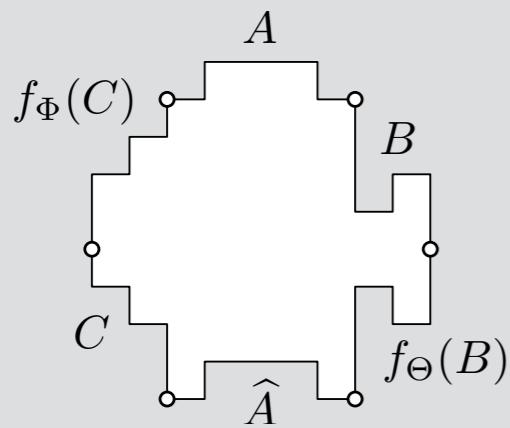
- [Gambini, Vuillon 2007]:  $O(n^2)$
- [Provençal 2008]:  $O(n * \log^3(n))$
- [Brlek, Provençal, Fédou 2009]:  $O(n)$  (special cases)
- [W. 2015]:  $O(n)$

# Algorithm

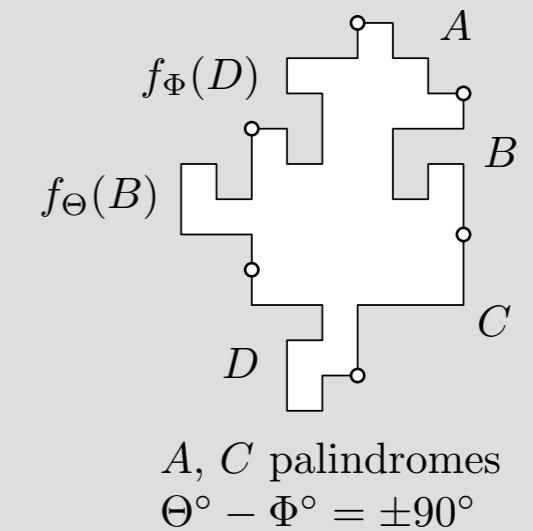
Test the input boundary for each criterion.



$B, C, D, E$  palindromes    $A, \hat{B}$  90-dromes,  $C$  palindrome



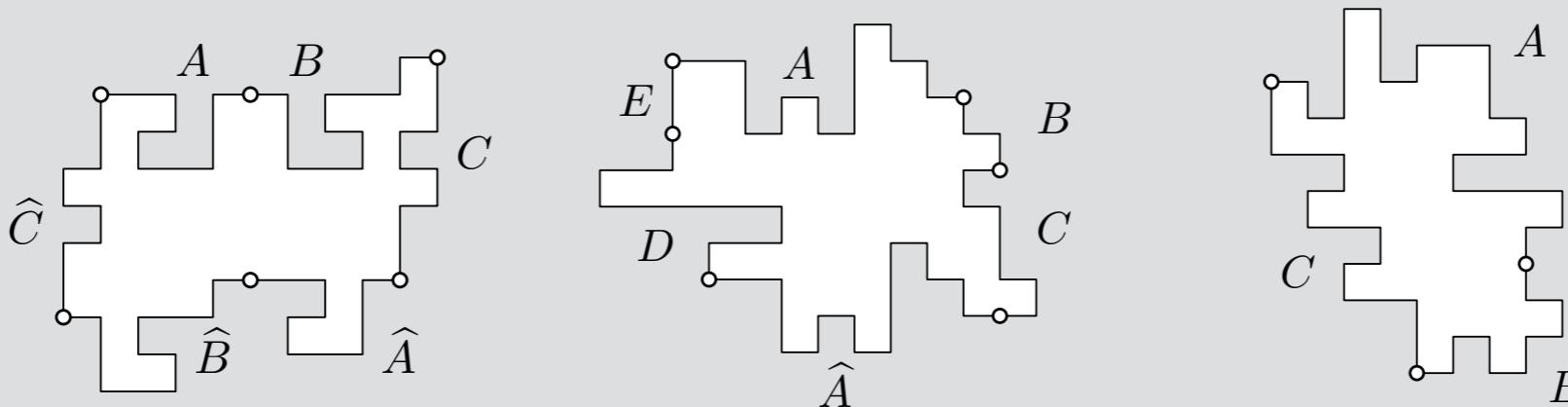
$B, C$  palindromes



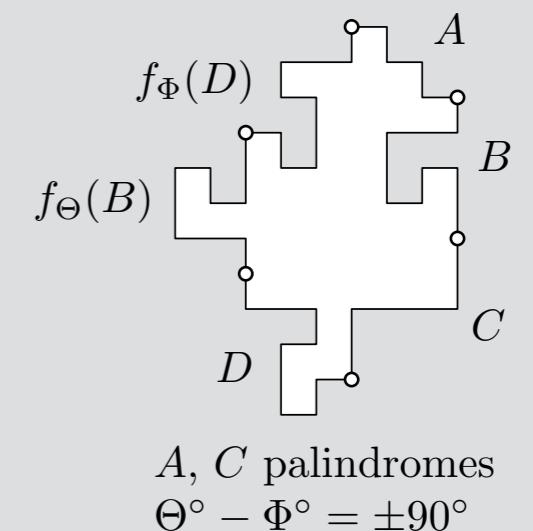
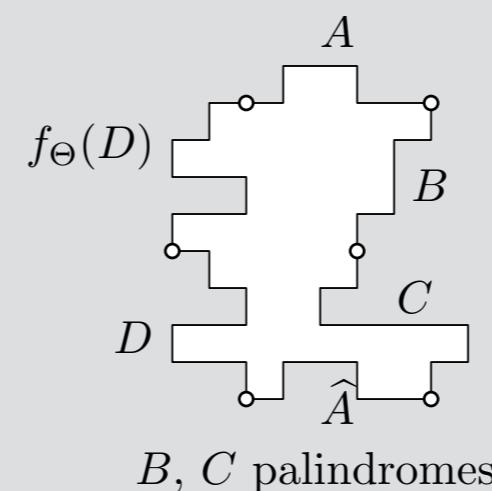
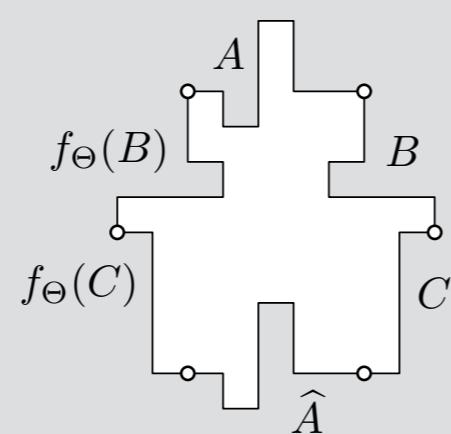
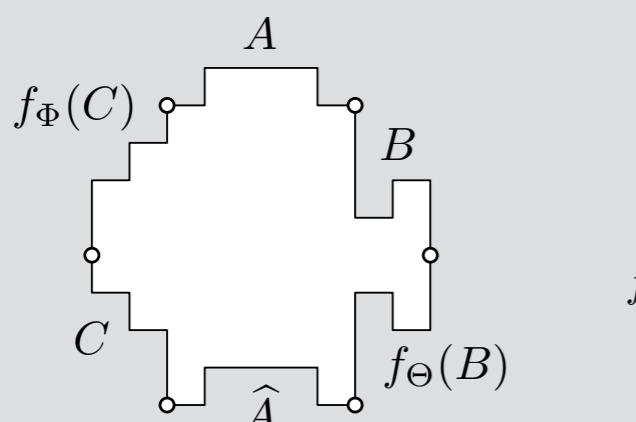
$A, C$  palindromes  
 $\Theta^\circ - \Phi^\circ = \pm 90^\circ$

Fast via several technical lemmas on words.

# Algorithm running times



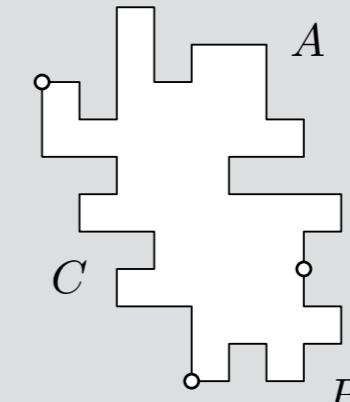
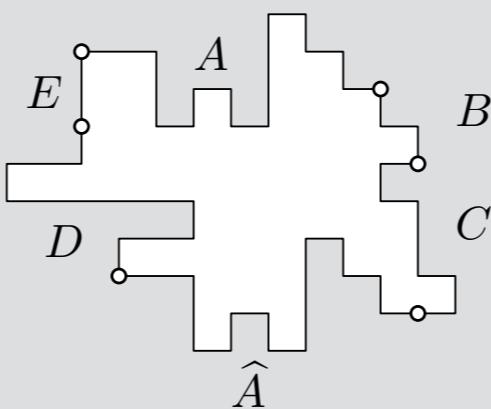
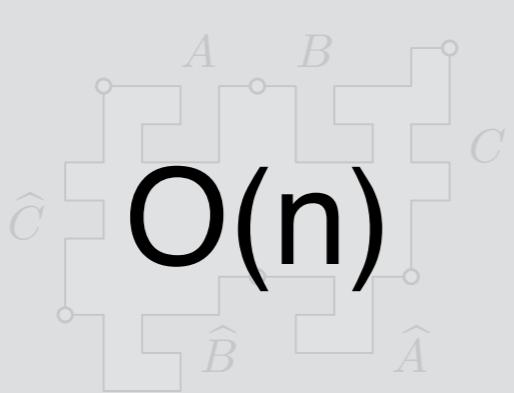
$B, C, D, E$  palindromes    $A$ ,  $B$  90-dromes,  $C$  palindrome



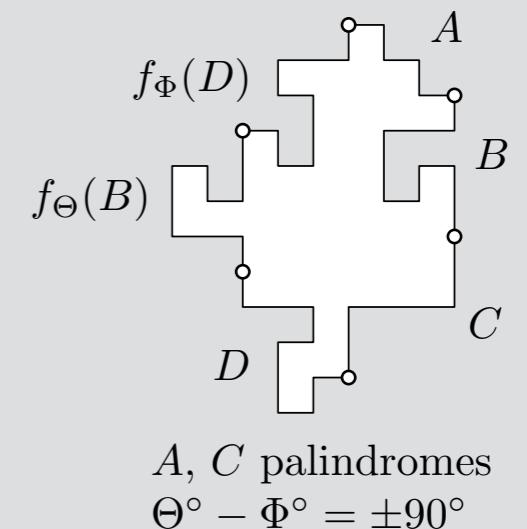
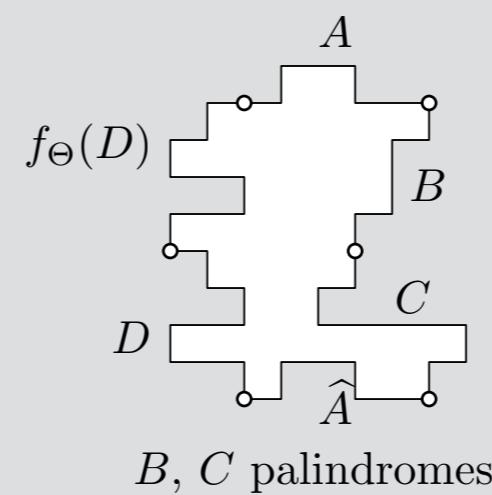
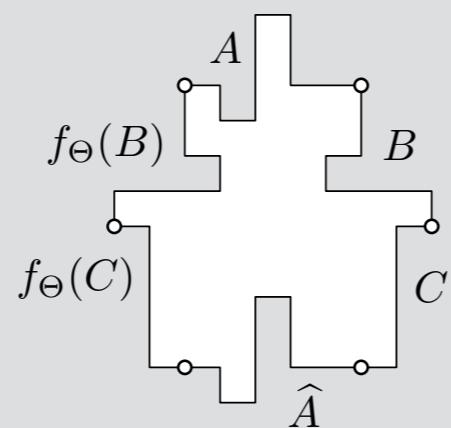
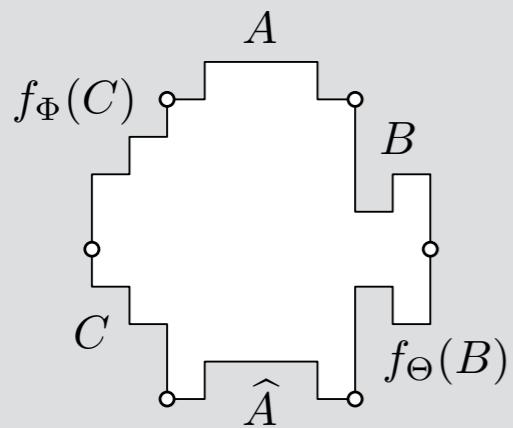
$B, C$  palindromes

$A, C$  palindromes  
 $\Theta^\circ - \Phi^\circ = \pm 90^\circ$

# Algorithm running times

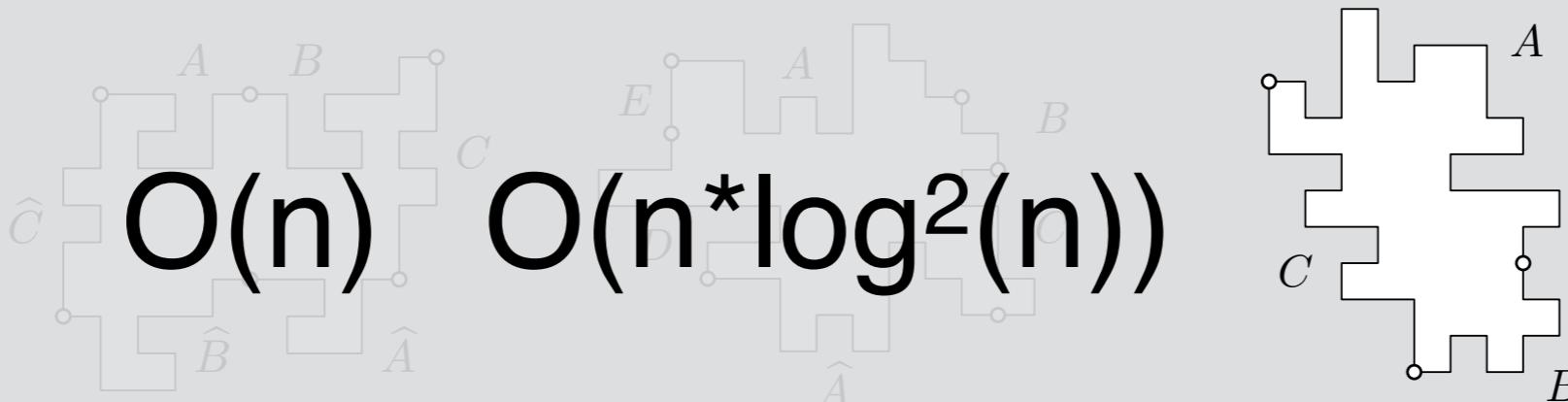


$B, C, D, E$  palindromes    $A, B$  90-dromes,  $C$  palindrome

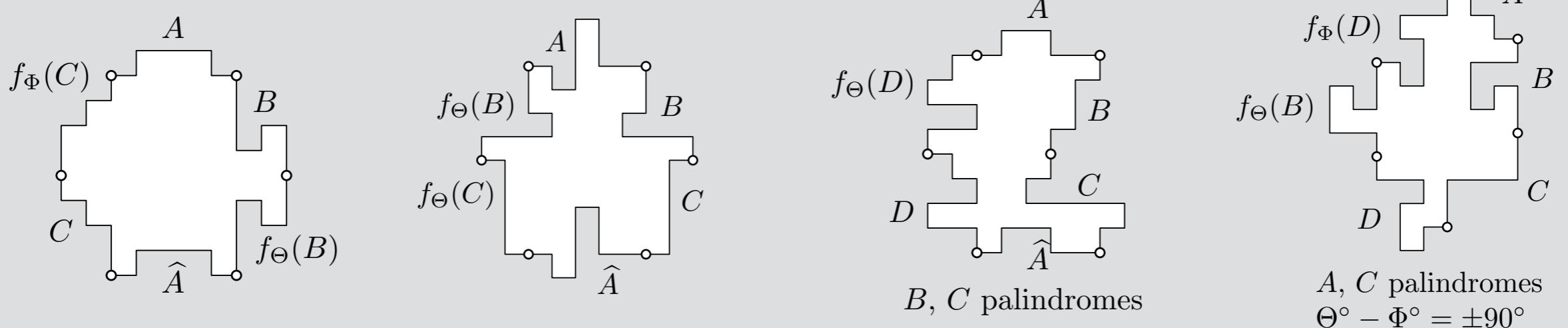


$A, C$  palindromes  
 $\Theta^\circ - \Phi^\circ = \pm 90^\circ$

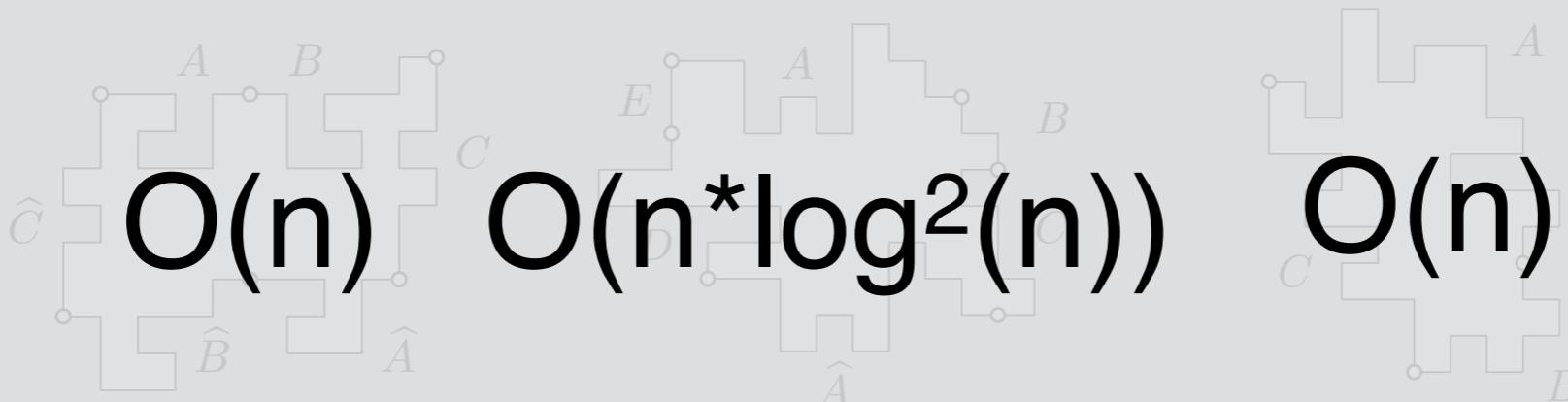
# Algorithm running times



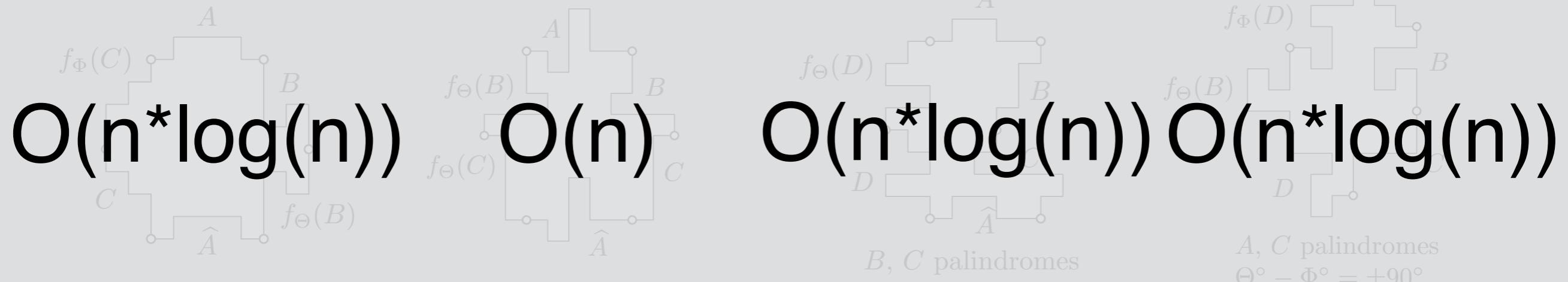
$B, C, D, E$  palindromes     $A, B$  90-dromes,  $C$  palindrome



# Algorithm running times



$B, C, D, E$  palindromes    $A, B$  90-dromes,  $C$  palindrome



$B, C$  palindromes

$A, C$  palindromes  
 $\Theta^\circ - \Phi^\circ = \pm 90^\circ$

# Algorithm running times

$O(n^* \log^2(n))$  total time

$B, C$  palindromes

$A, C$  palindromes  
 $\Theta^\circ = \Phi^\circ = \pm 90^\circ$

# Open Problems

Theorem:  $O(n^* \log^2(n))$ -time algorithm for deciding if a polyomino tiles the plane isohedrally.

# Open Problems

Theorem:  $O(n^* \log^2(n))$ -time algorithm for deciding if a polyomino tiles the plane isohedrally.

$O(n)$ -time algorithm?

# Open Problems

Theorem:  $O(n^* \log^2(n))$ -time algorithm for deciding if a polyomino tiles the plane isohedrally.

$O(n)$ -time algorithm?

Enumeration of tilings in  $O(n^* \log^2(n) + t)$  time?

# Open Problems

Theorem:  $O(n^* \log^2(n))$ -time algorithm for deciding if a polyomino tiles the plane isohedrally.

$O(n)$ -time algorithm?

Enumeration of tilings in  $O(n^* \log^2(n) + t)$  time?

Extend inputs to polygons?

# Open Problems

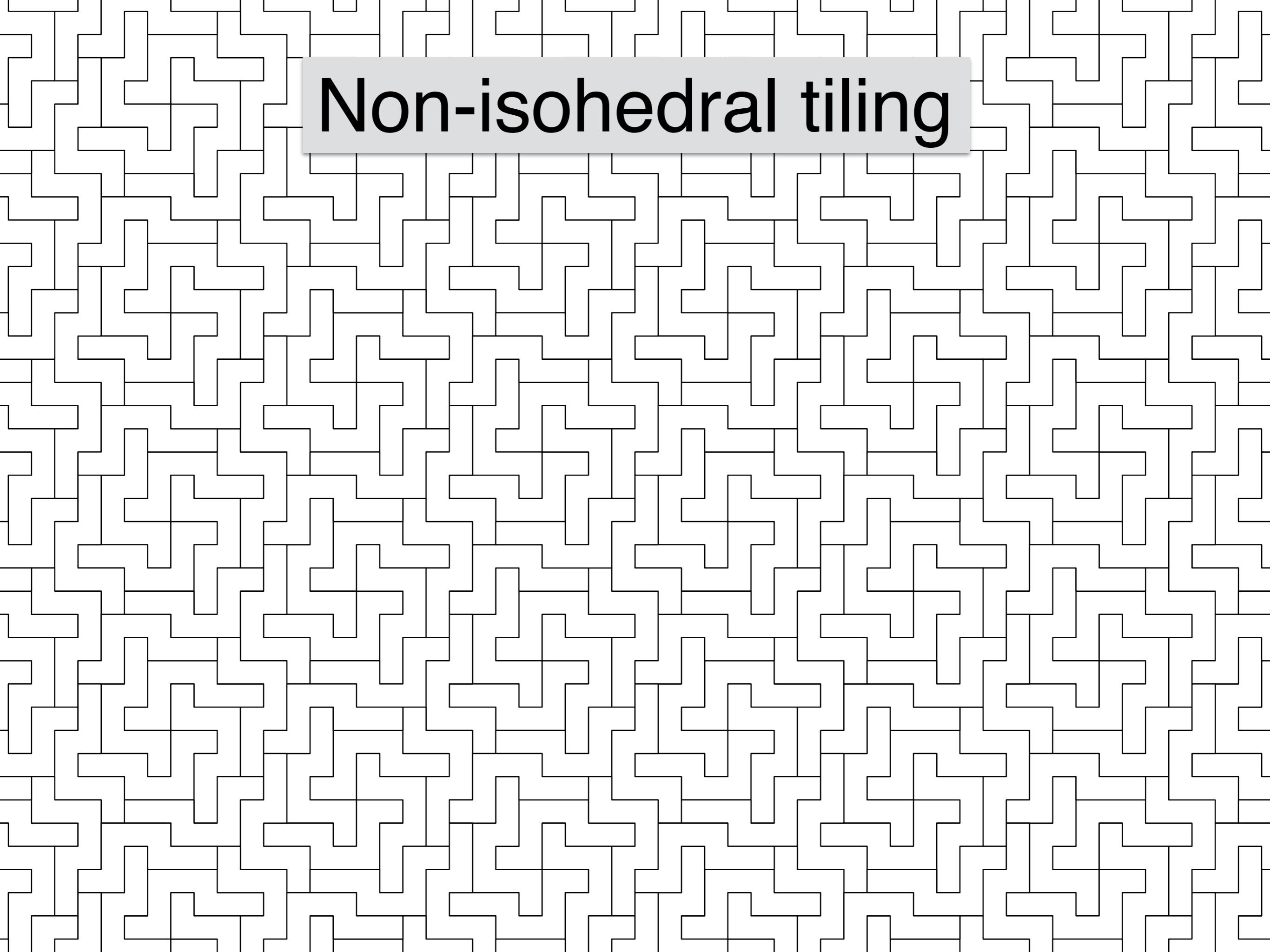
Theorem:  $O(n^* \log^2(n))$ -time algorithm for deciding if a polyomino tiles the plane **isohedrally**.

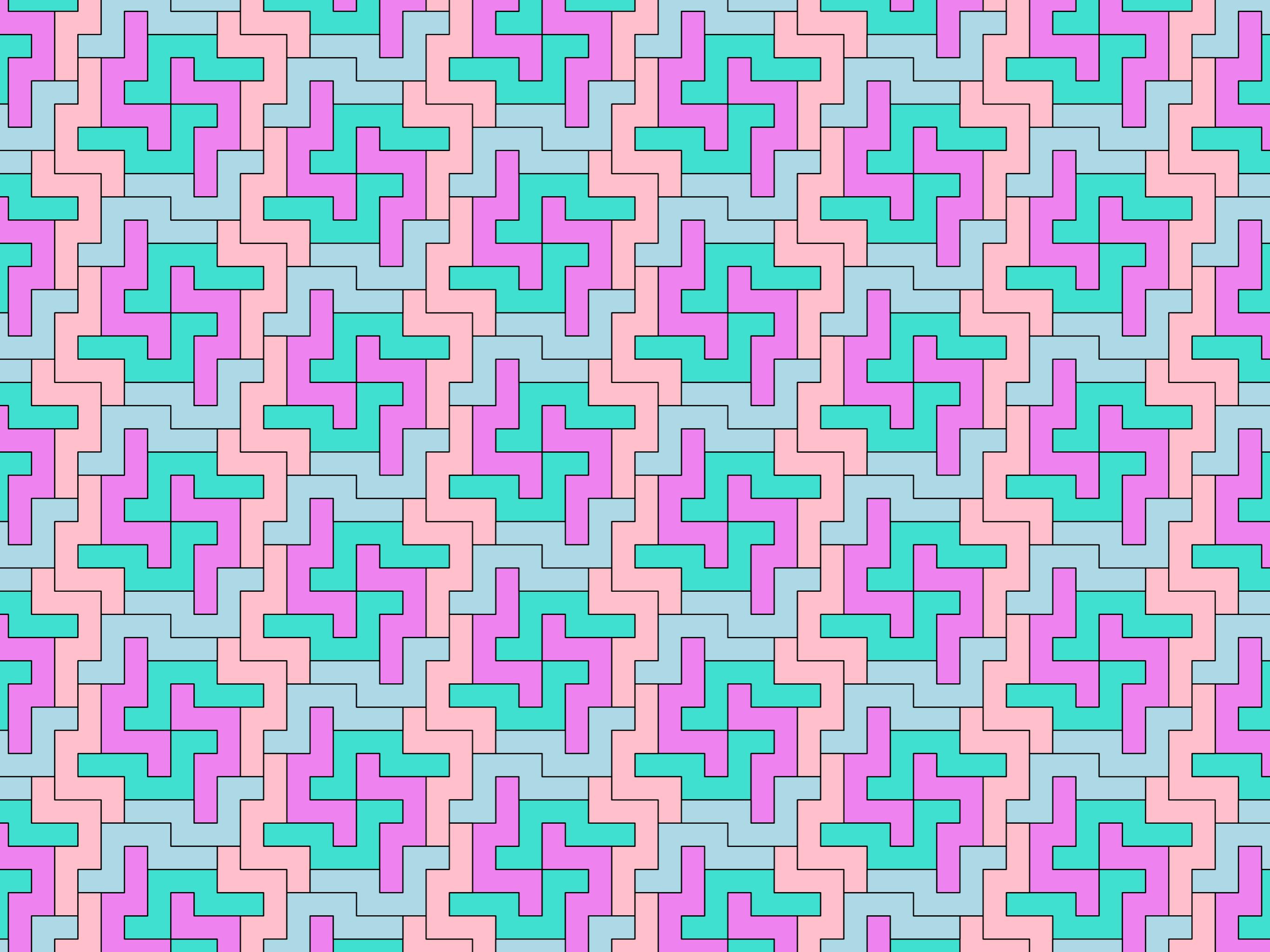
$O(n)$ -time algorithm?

Enumeration of tilings in  $O(n^* \log^2(n) + t)$  time?

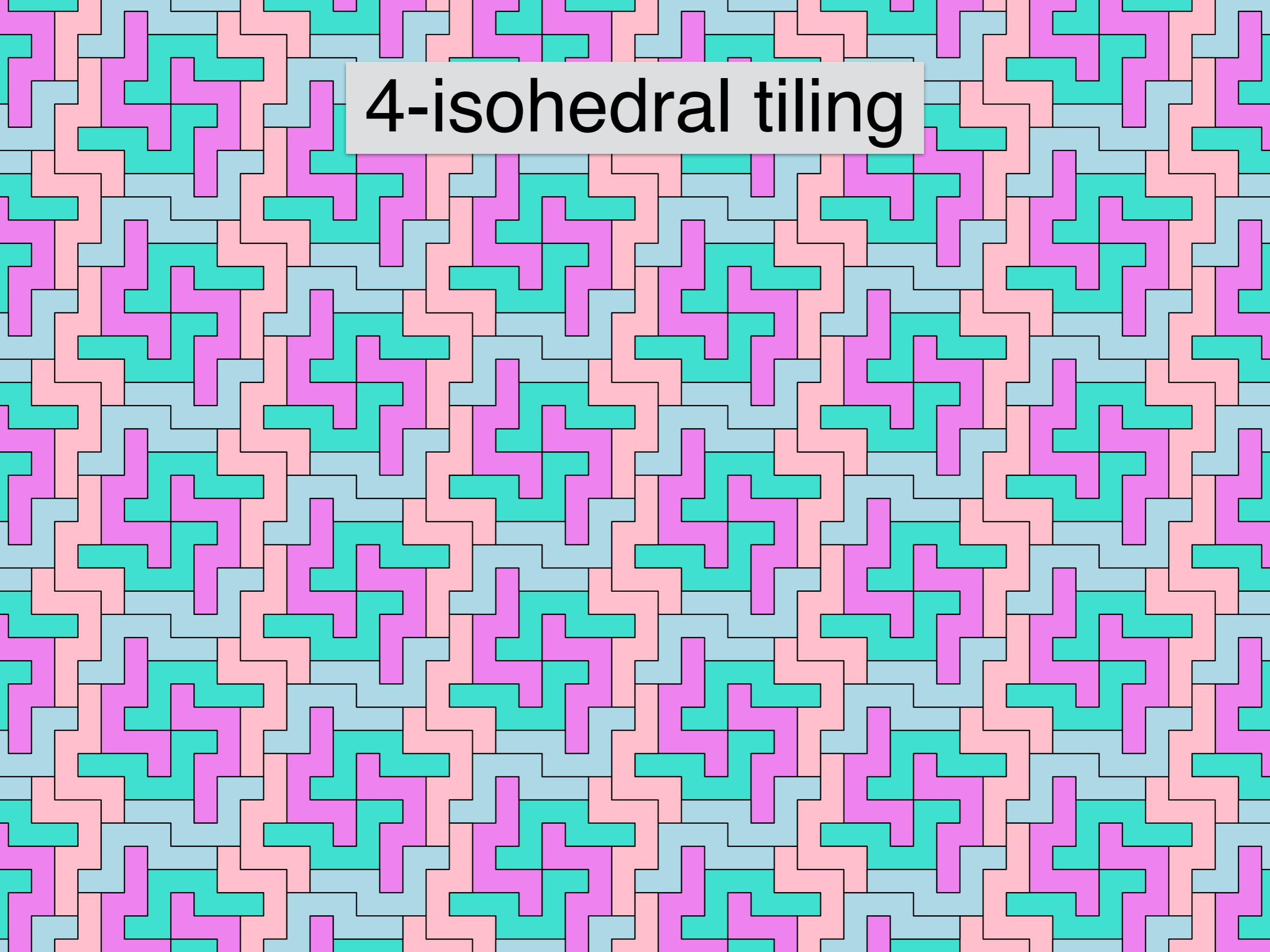
Extend inputs to polygons?

# Non-isohedral tiling



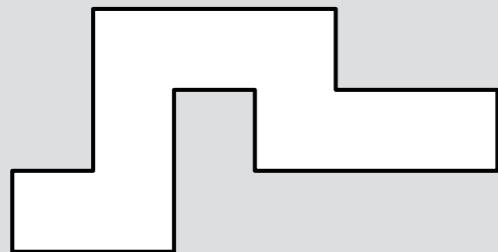


# 4-isohedral tiling

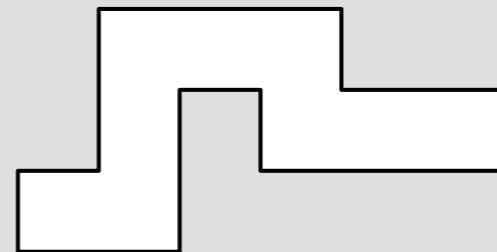


# Problem

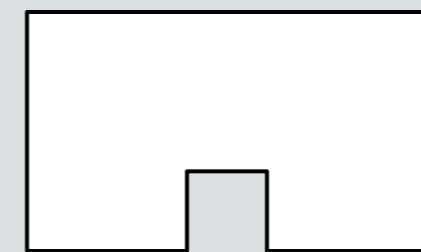
Decide whether a polyomino  
has a  $\leq k$ -isohedral tiling.



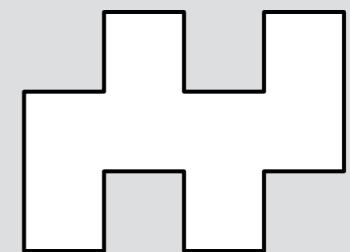
$k = 5$



$k = 3$



$k = 3$



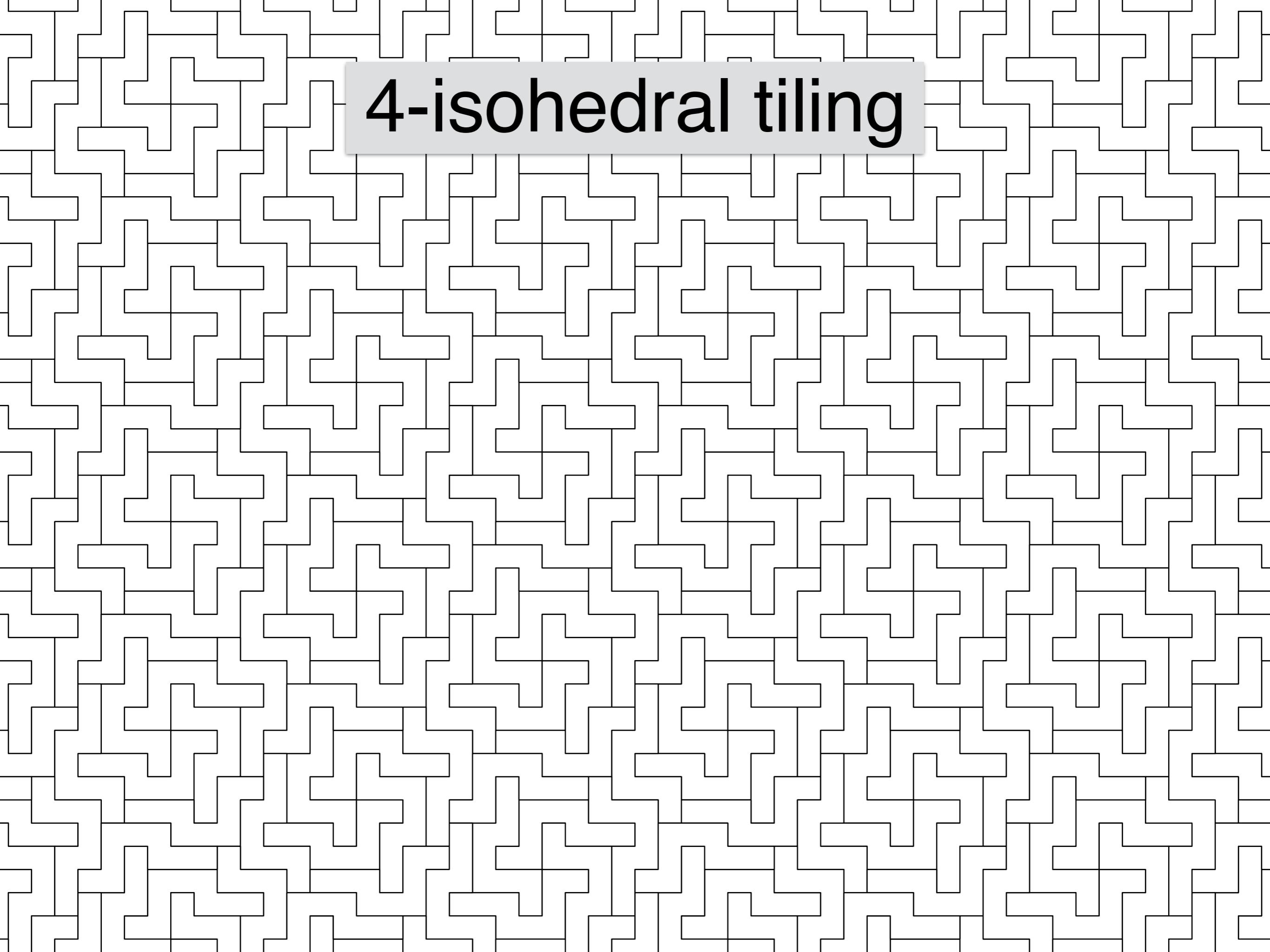
$k = 1$

Yes

No

No

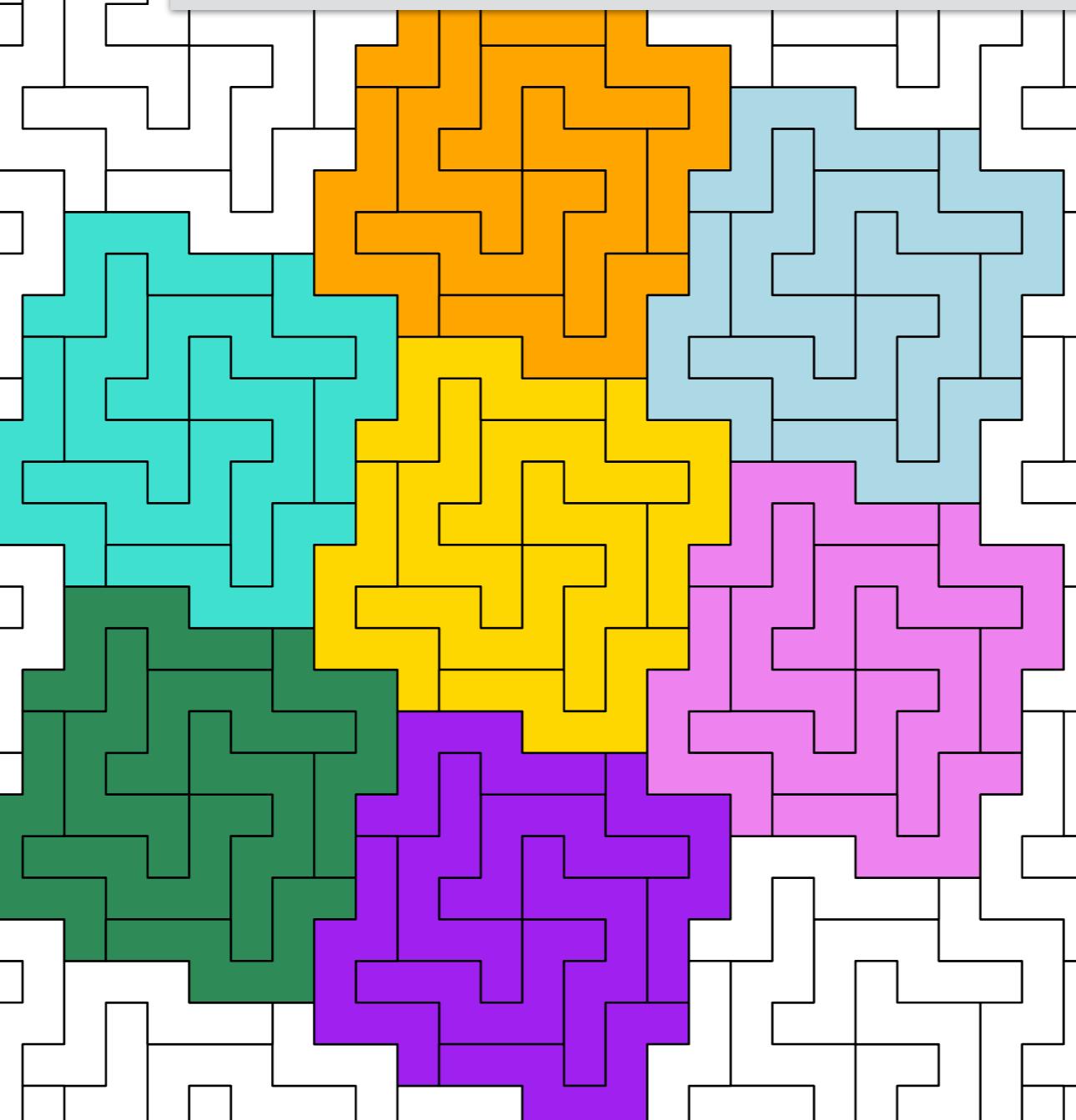
Yes



# 4-isohedral tiling

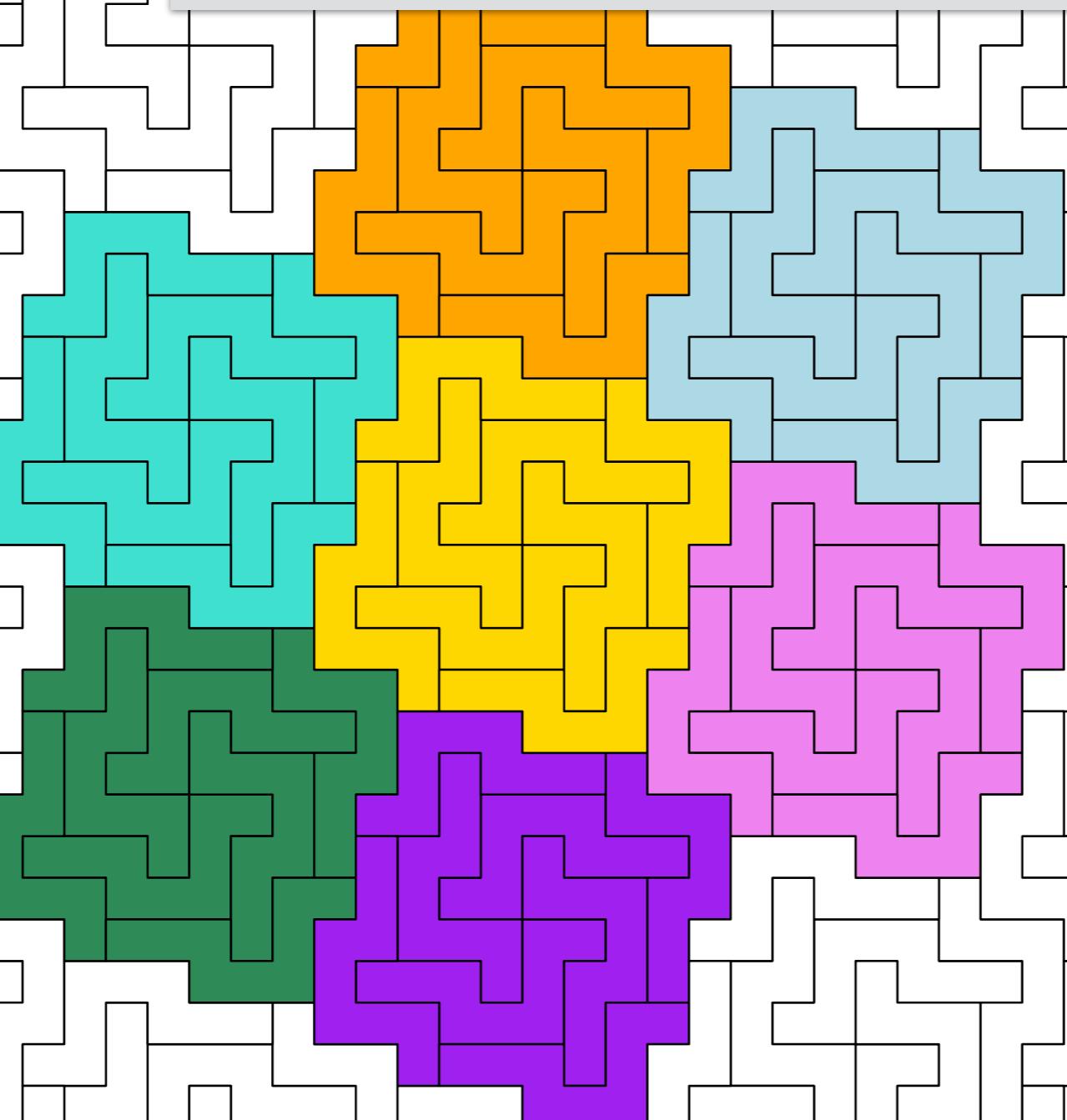
# 4-isohedral tiling

Fundamental  
domain

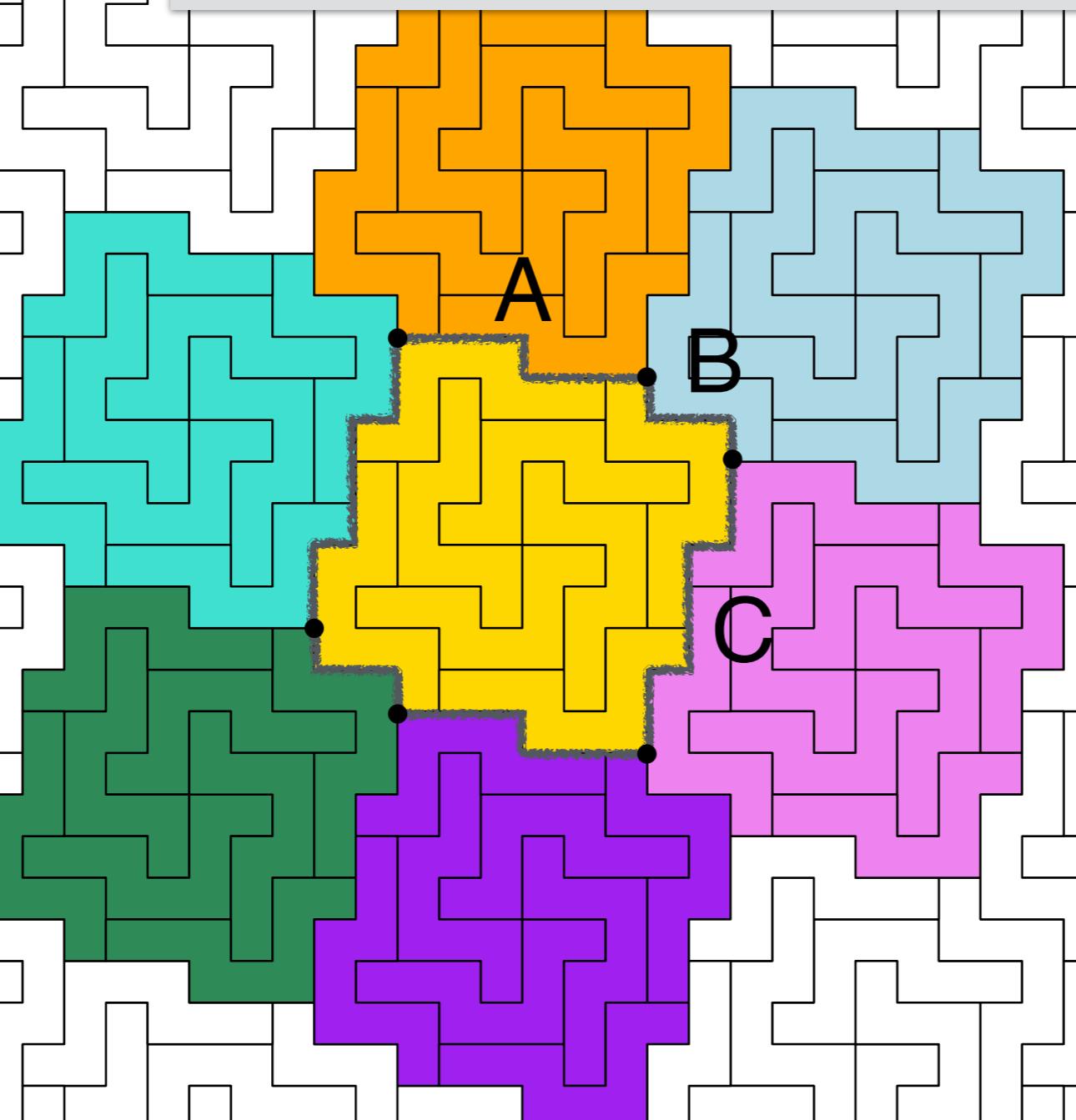


# 4-isohedral tiling

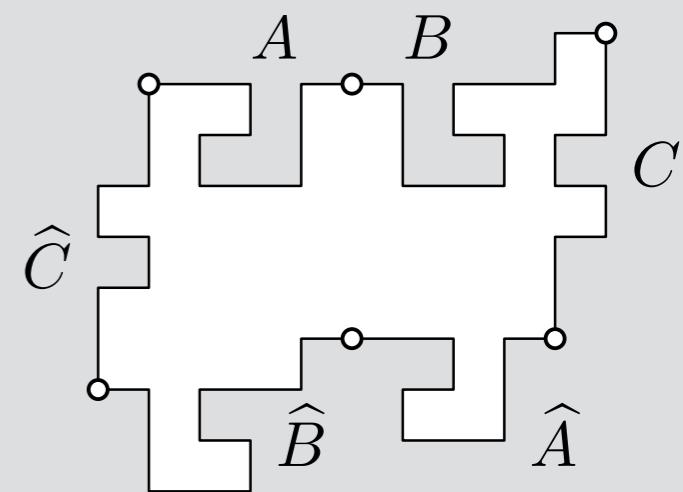
Fundamental  
domain  
of size  $\leq 4^*8$



# 4-isohedral tiling



Fundamental  
domain  
of size  $\leq 4^*8$   
satisfies



translation criterion

# Problem

Decide whether a polyomino  
has a  $\leq k$ -isohedral tiling.

# Algorithm

Check every fundamental domain  
of size  $\leq k^*8$  for translation criterion.

# Problem

Decide whether a polyomino  
has a  $\leq k$ -isohedral tiling.

# Algorithm

Check every fundamental domain  
 $n^{O(k)}$ -time  
of size  $\leq k^8$  for translation criterion.

Is there an  $f(k)n^{O(1)}$ -time algorithm?

# 50-year-old Problem

Decide whether a polyomino  
has a  ~~$\leq k$ -isohedral~~ tiling.

Algorithm

$n^{O(k)}$ -time

Is there an  ~~$f(k)n^{O(1)}$ -time~~ algorithm?

# A Quasilinear-Time Algorithm for Tiling the Plane Isohedrally with a Polyomino

Stefan Langerman<sup>\*1</sup> and Andrew Winslow<sup>1</sup>

1 Département d’Informatique, Université Libre de Bruxelles,  
ULB CP212, boulevard du Triomphe, 1050 Bruxelles, Belgium,  
`{stefan.langerman, andrew.winslow}@ulb.ac.be`