A Quasilinear-Time Algorithm for Tiling the Plane Isohedrally with a Polyomino

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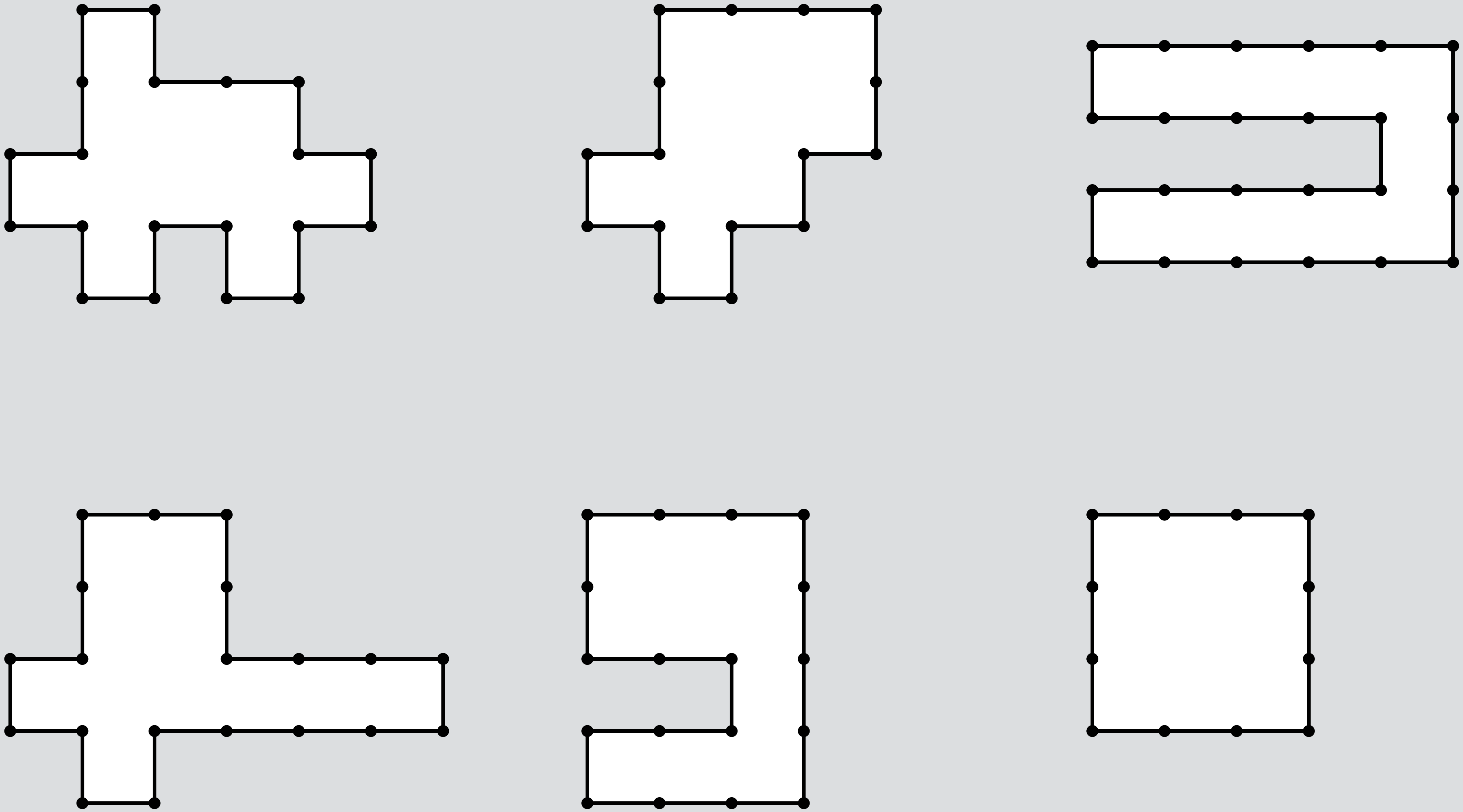
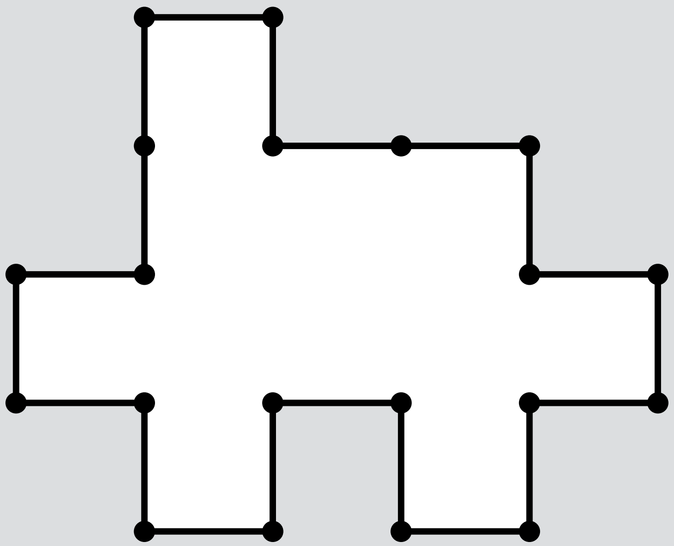
Polyominoes

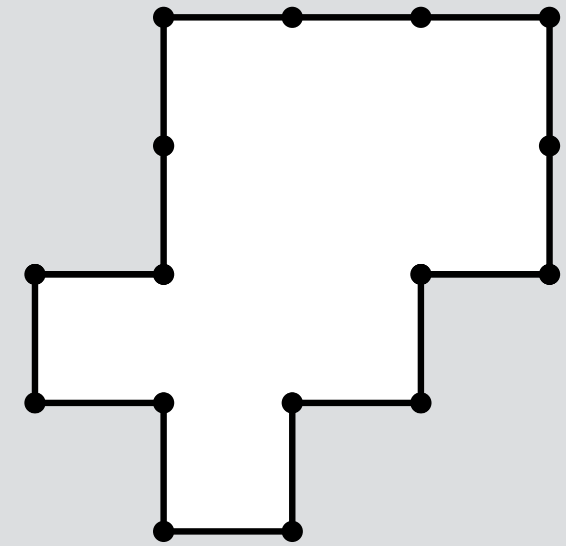
Rectilinear simple polygons with unit edge lengths



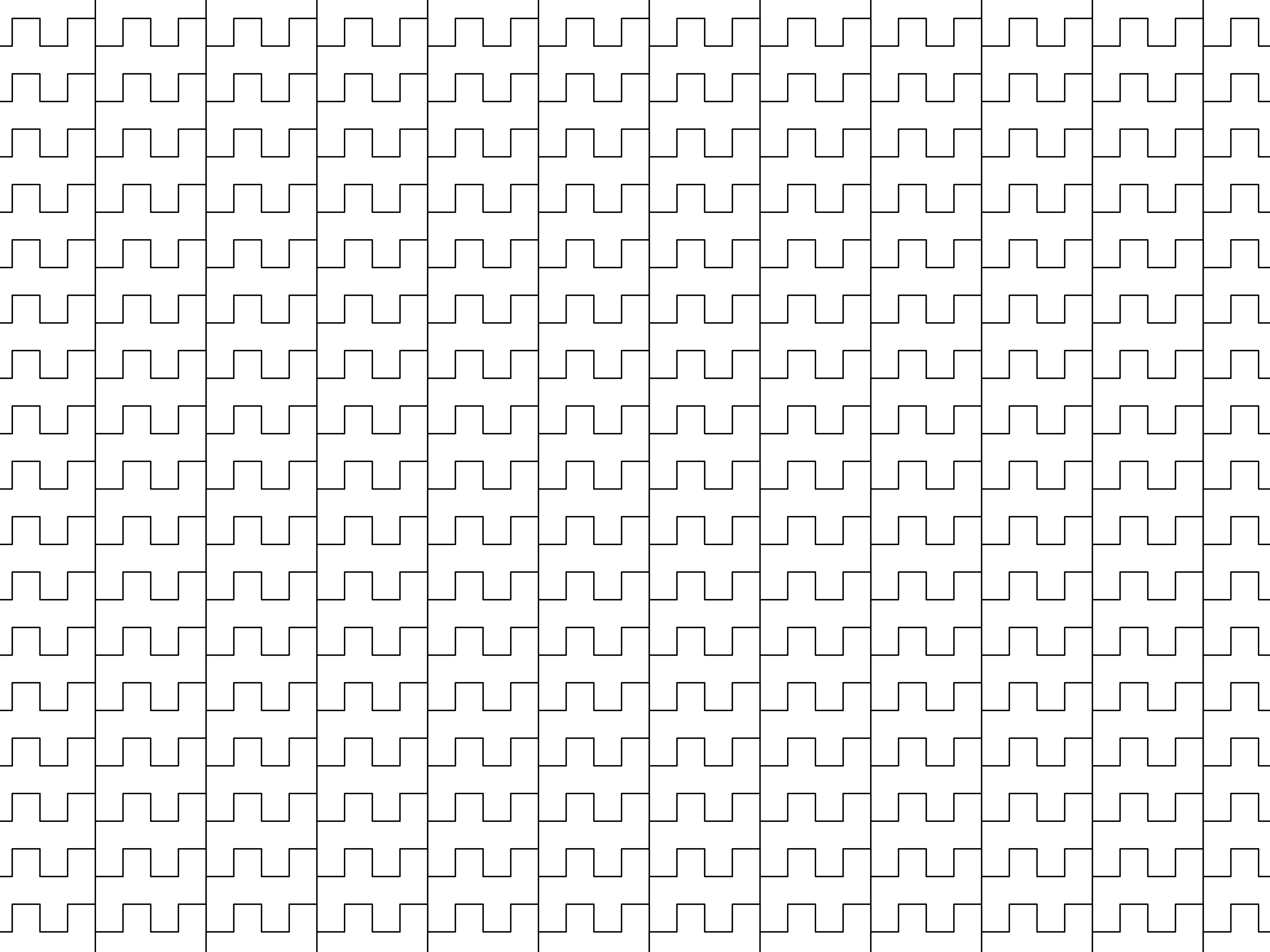
Boundary words



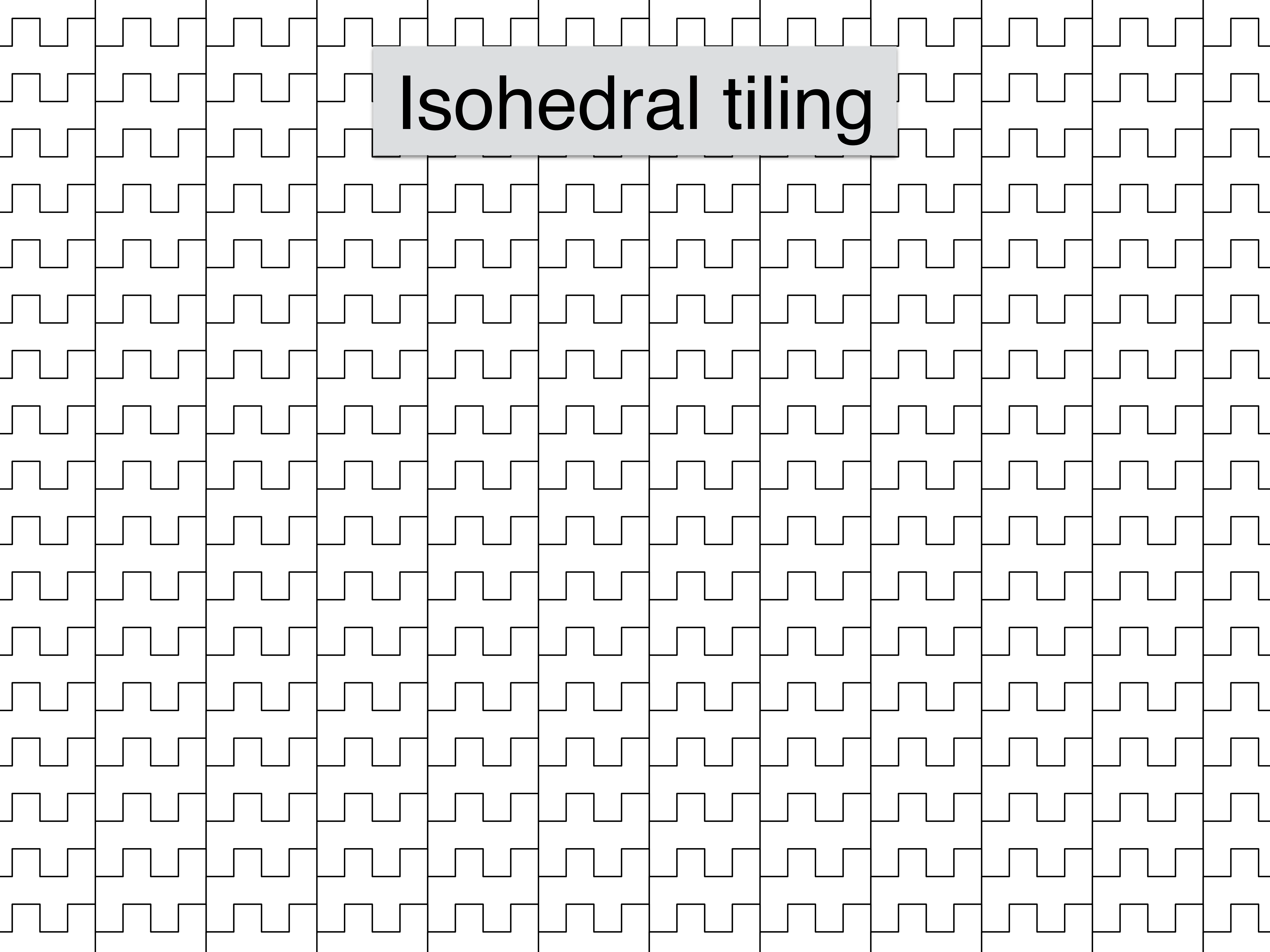
$$uru^2rdr^2drd(ldlu)^2l$$



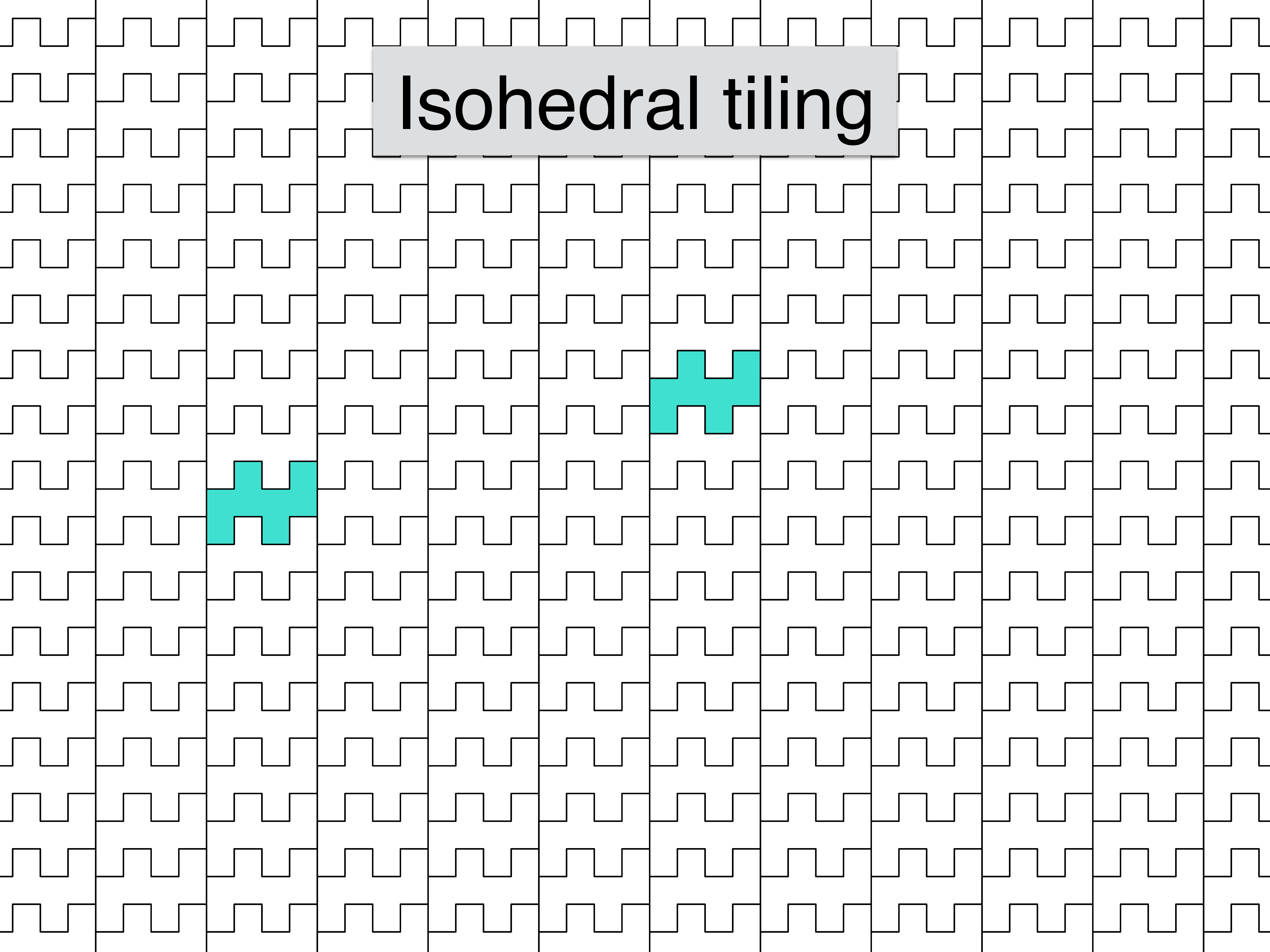
$$d(dl)^3uluru^2r^3$$



Isohedral tiling



Isohedral tiling



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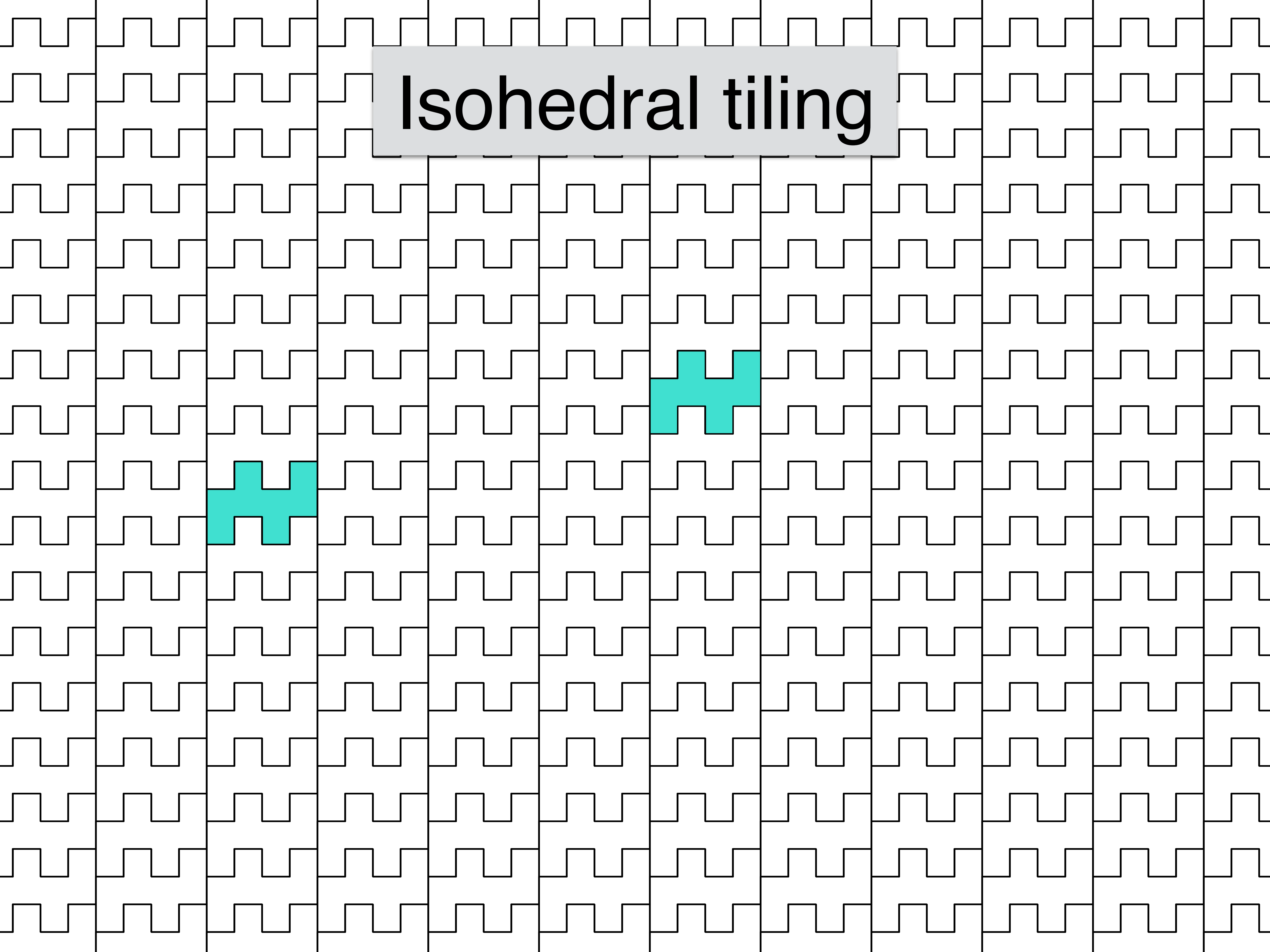

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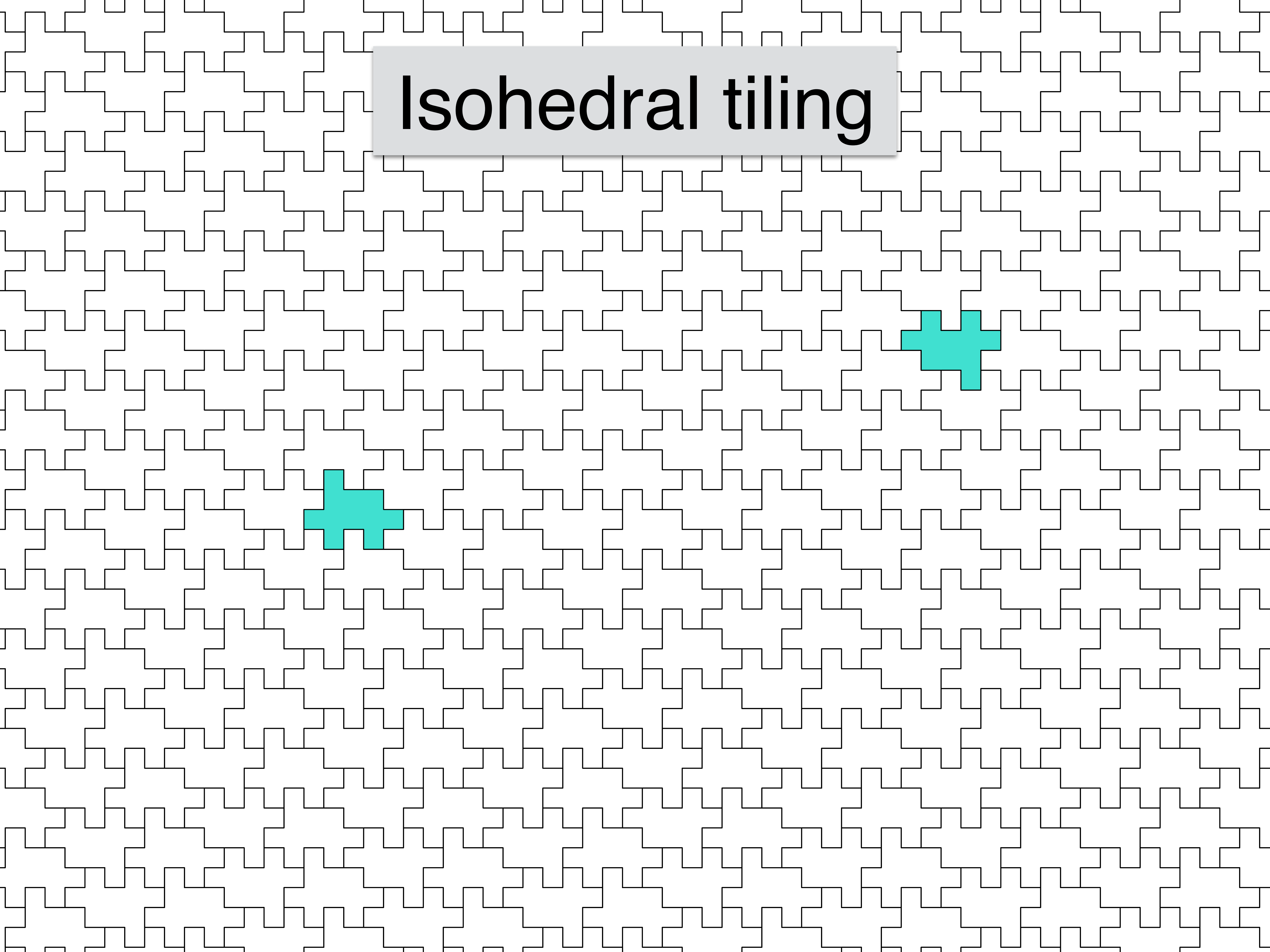

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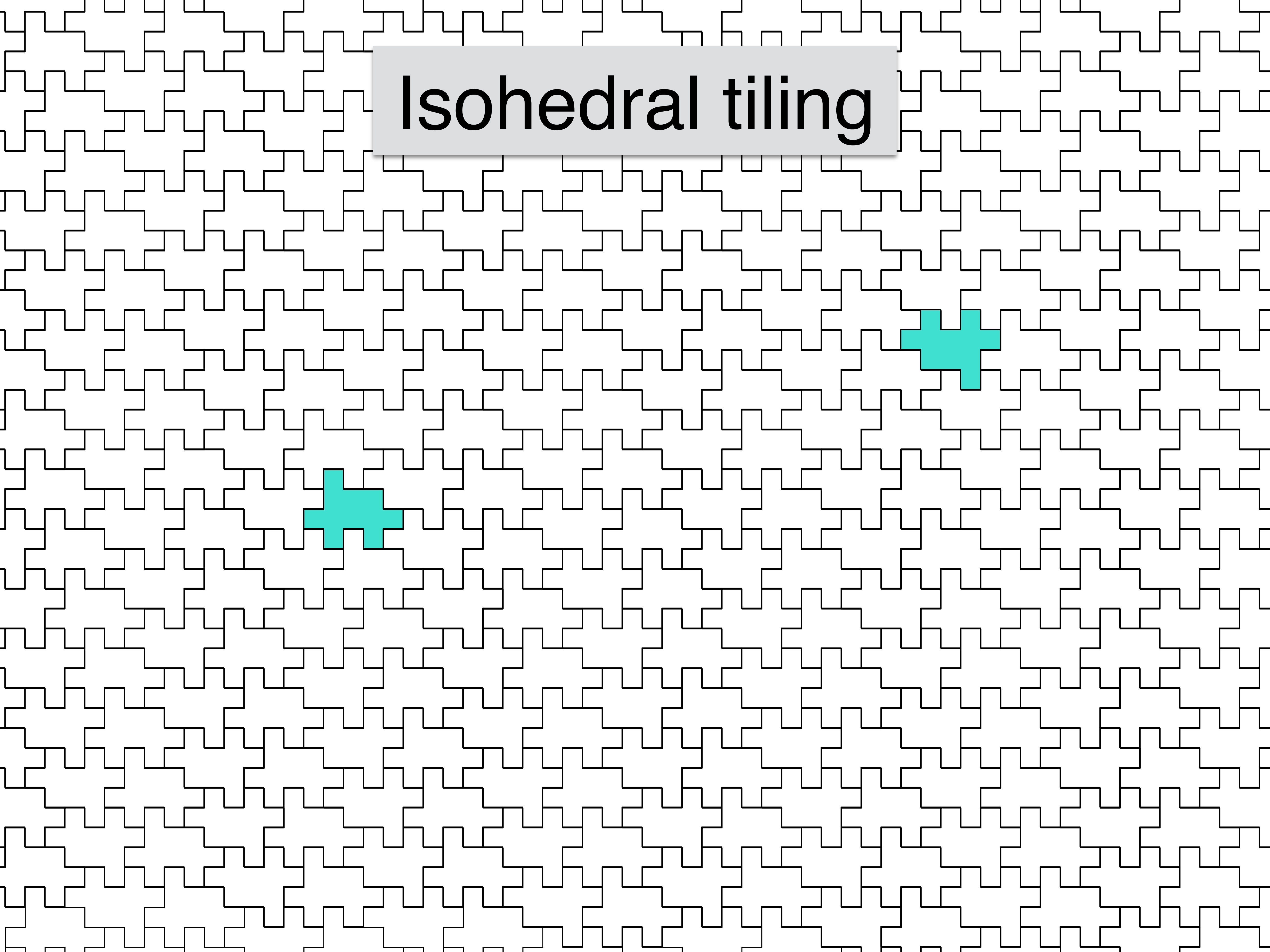
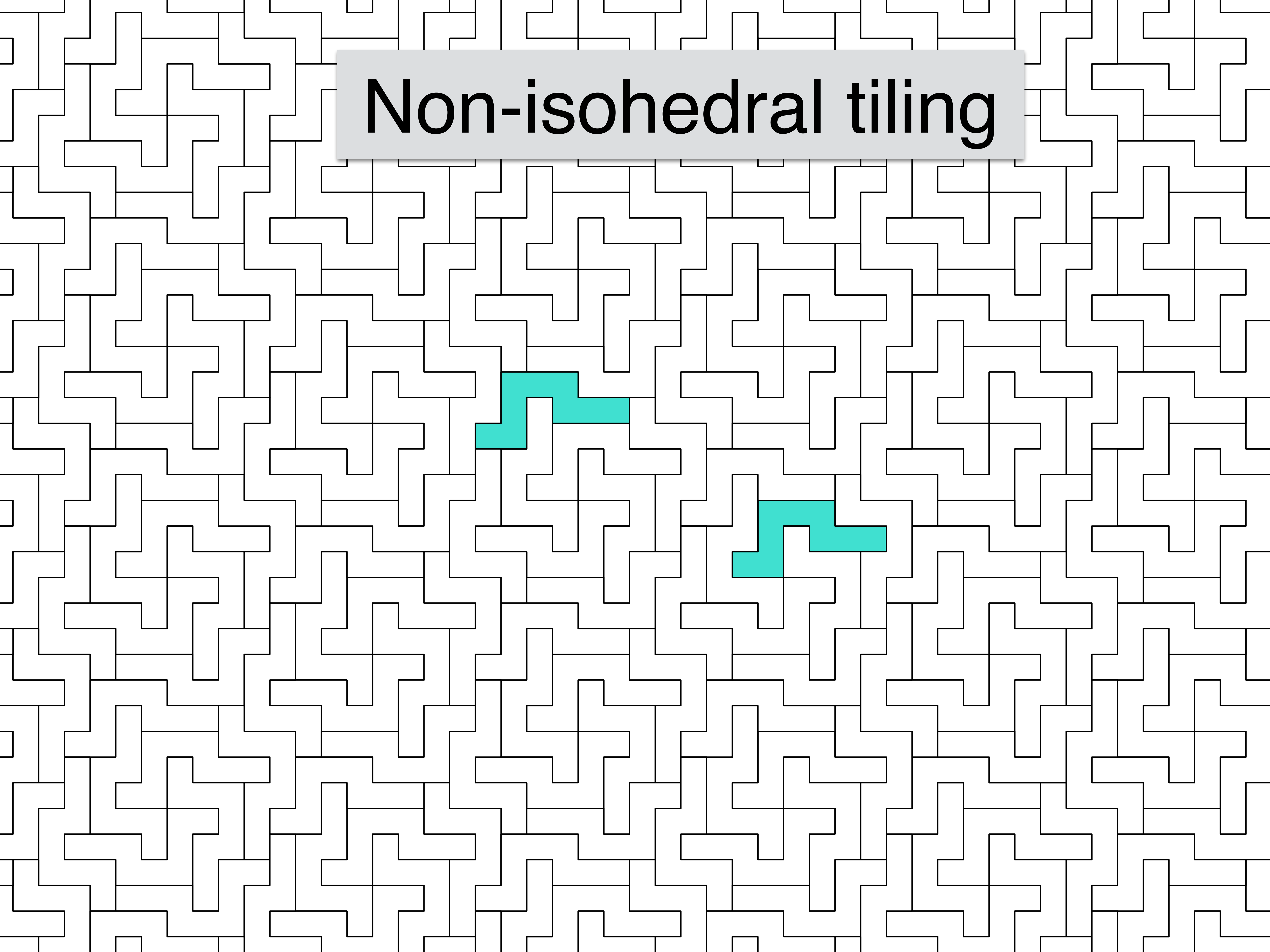
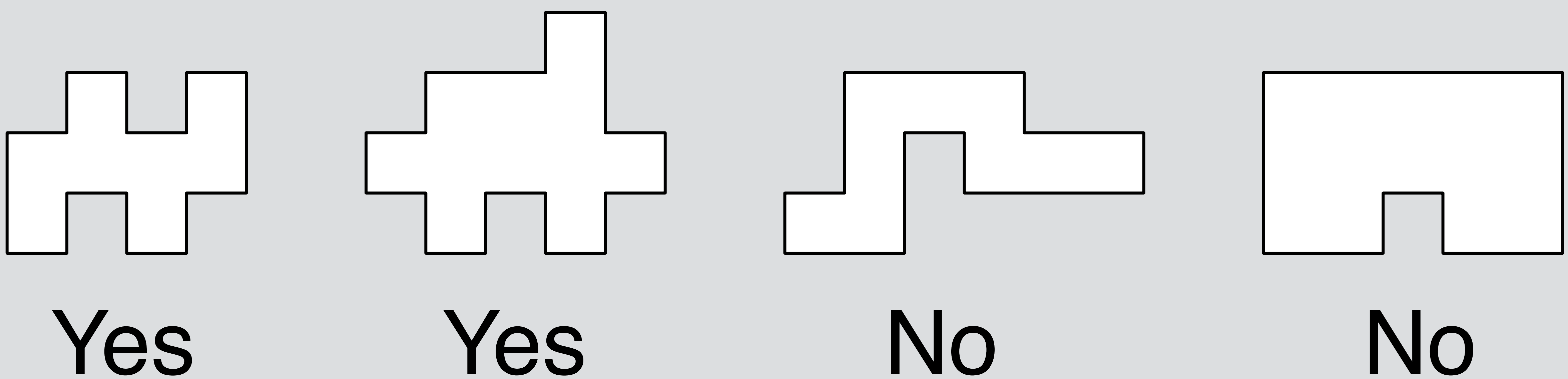


Non-isohedral tiling

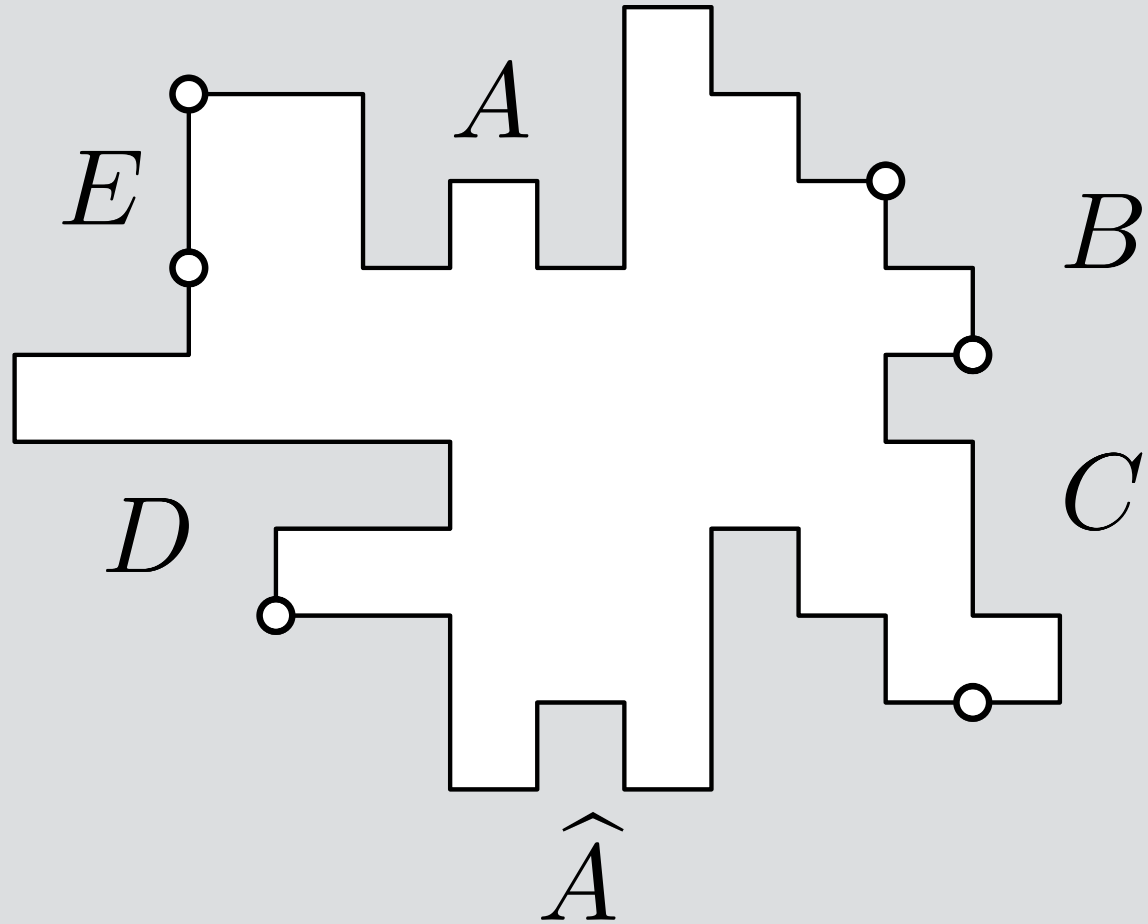


Problem

Decide whether a polyomino has an isohedral tiling.



Conway's criterion



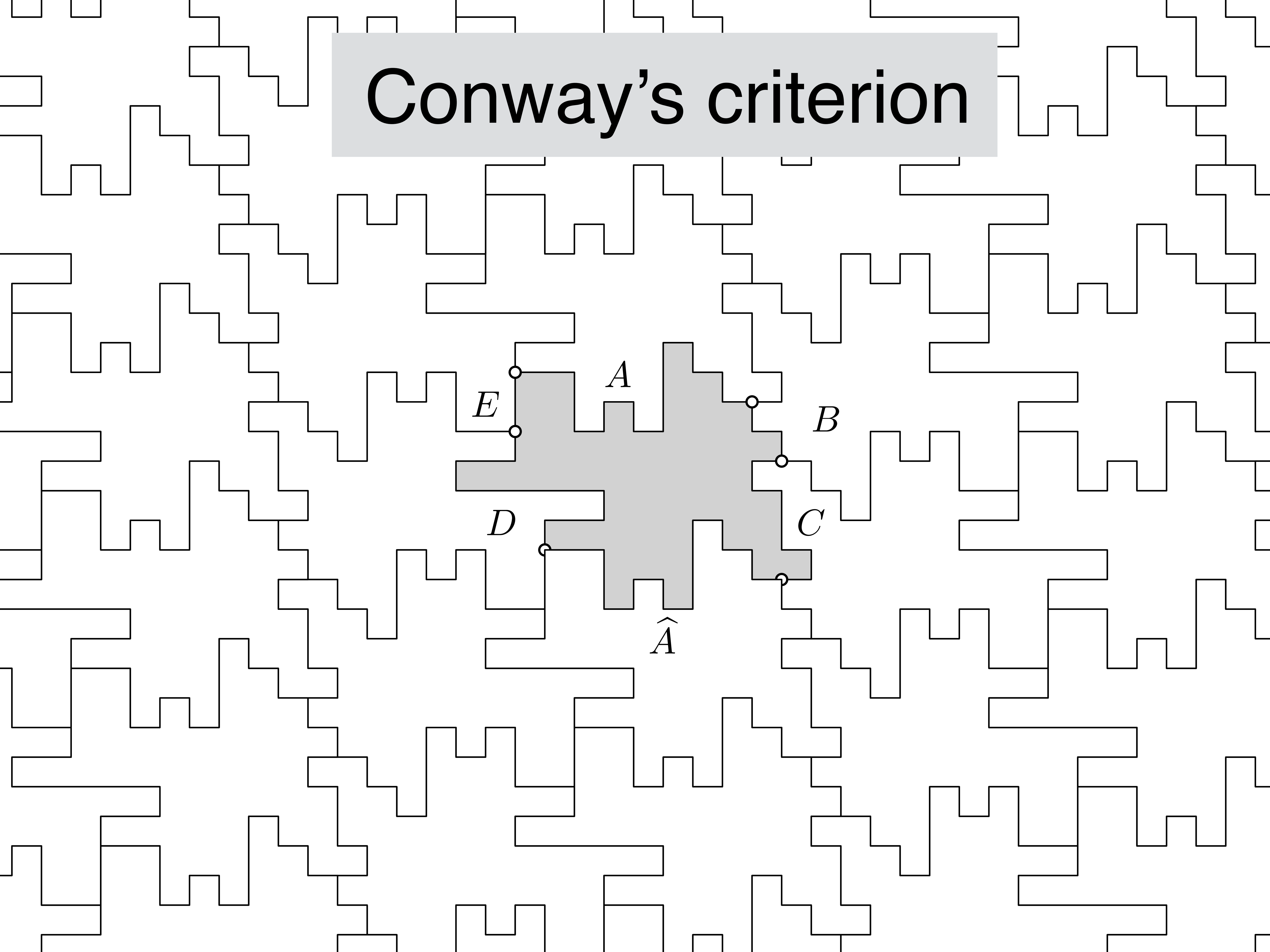
B, C, D, E palindromes

$$X = x_1 x_2 \ldots x_n$$
$$\widehat{X} = \overline{x}_n \overline{x}_{n-1} \ldots \overline{x}_1$$

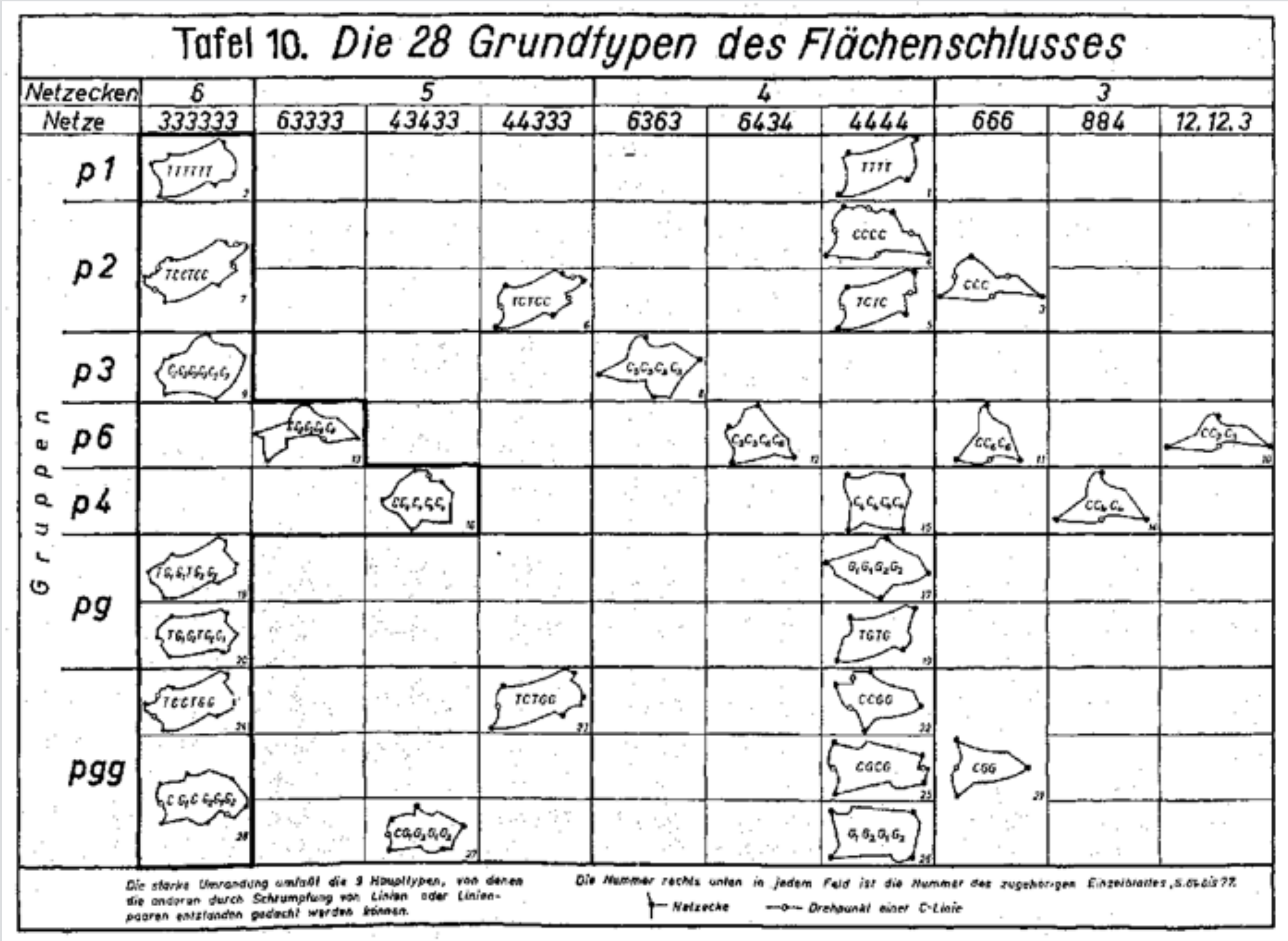
with

$$\overline{u} = d \quad \overline{r} = l$$
$$\overline{d} = u \quad \overline{l} = r$$

Conway's criterion

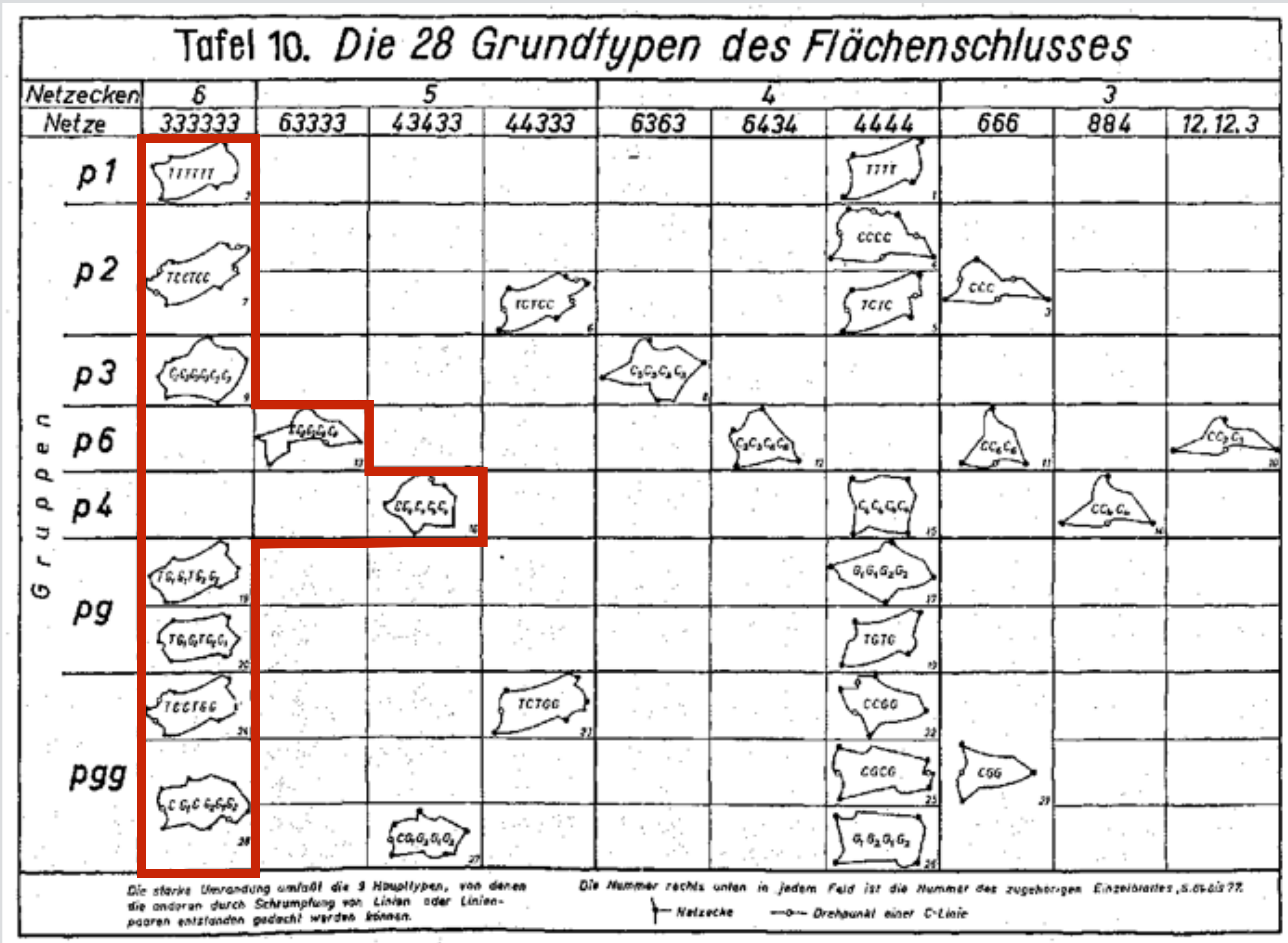


Isohedral boundary criteria



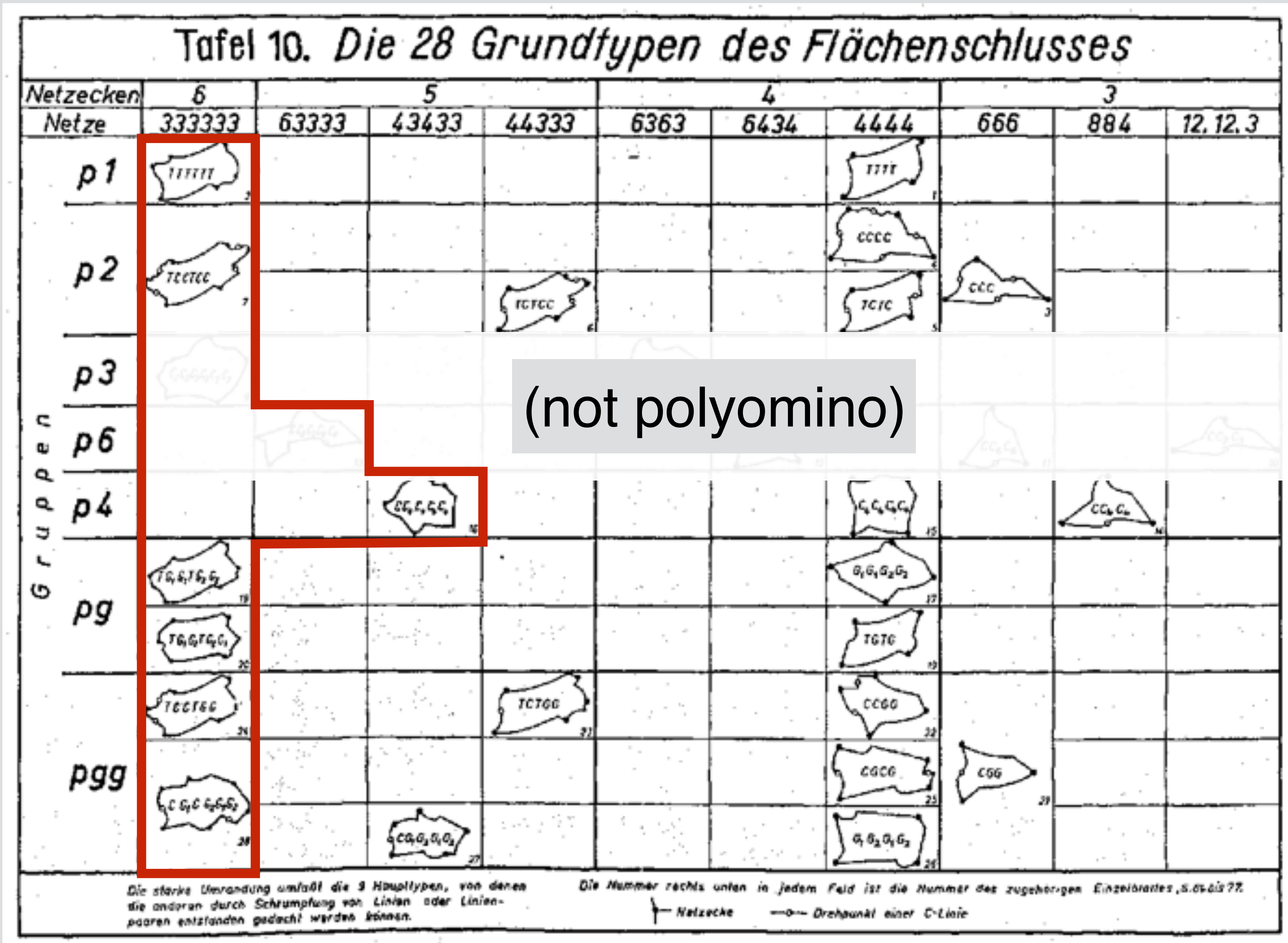
[Heesch, Kienzle 1963]

Isohedral boundary criteria



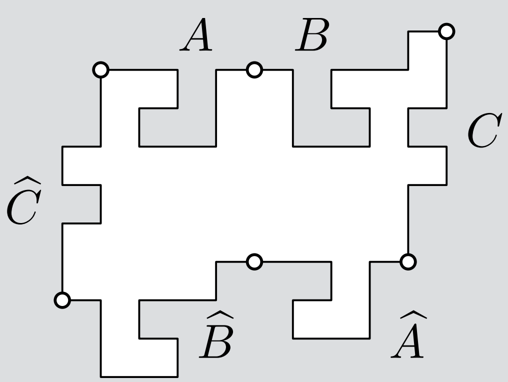
[Heesch, Kienzle 1963]

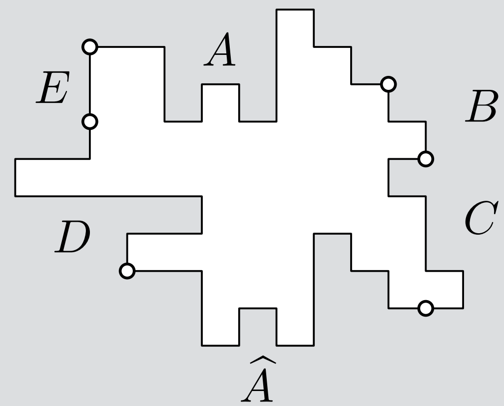
Isohedral boundary criteria



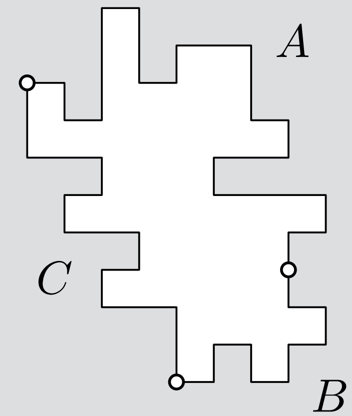
[Heesch, Kienzle 1963]

7 isohedral criteria

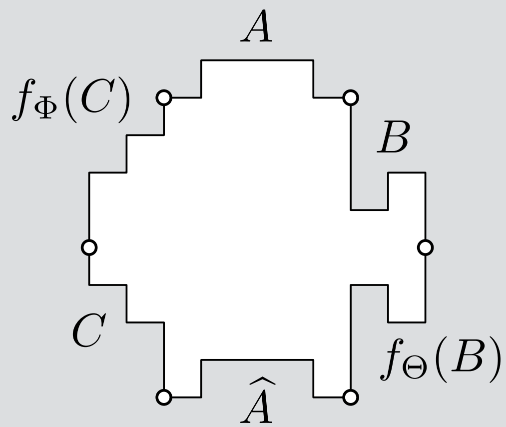


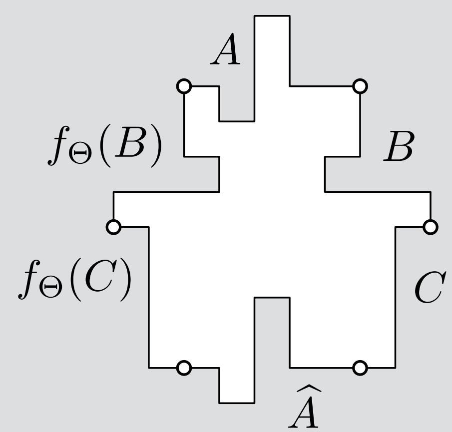


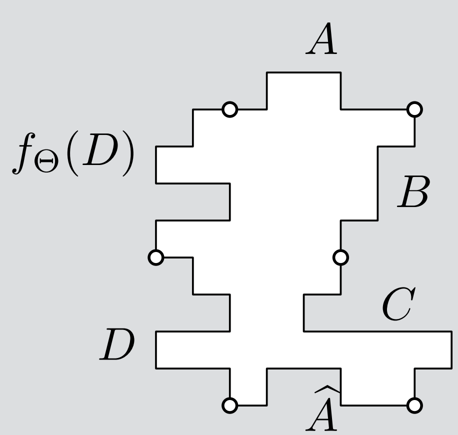
B, C, D, E palindromes



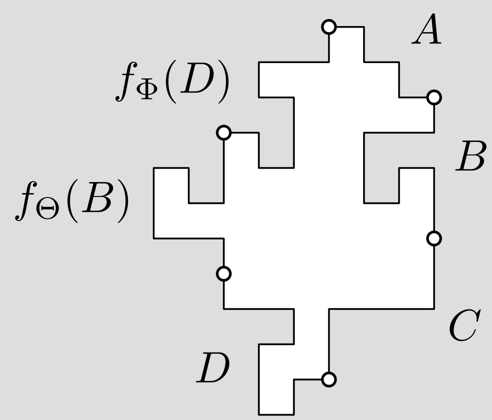
A, B 90-dromes, C palindrome







B, C palindromes



A, C palindromes
$\Theta^\circ - \Phi^\circ = \pm 90^\circ$

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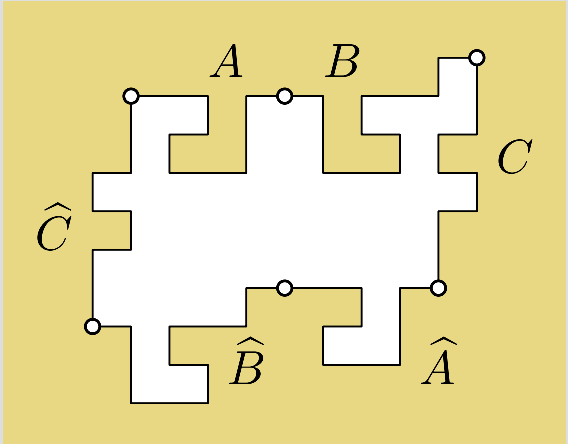
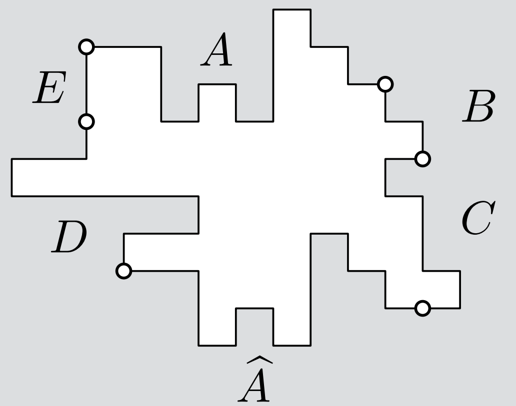


Translation criterion



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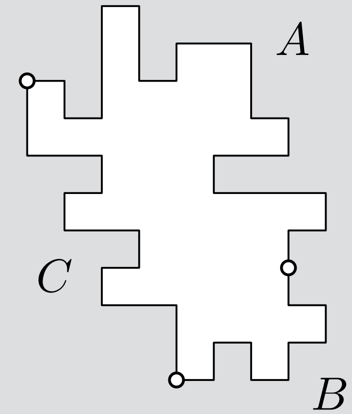

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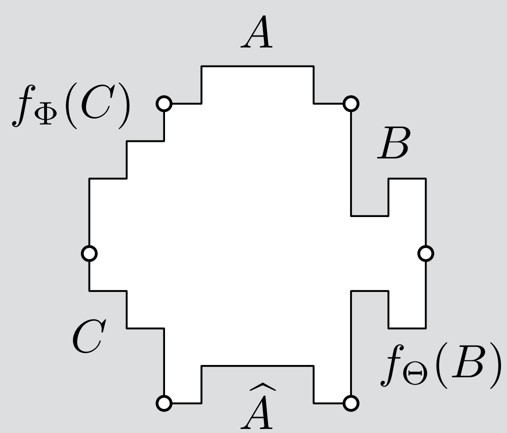

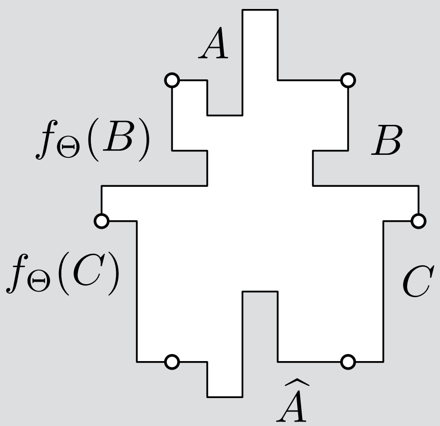

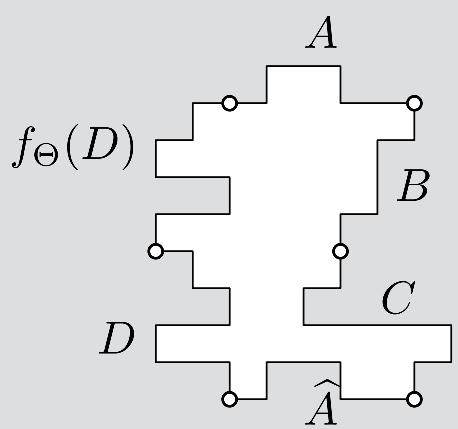

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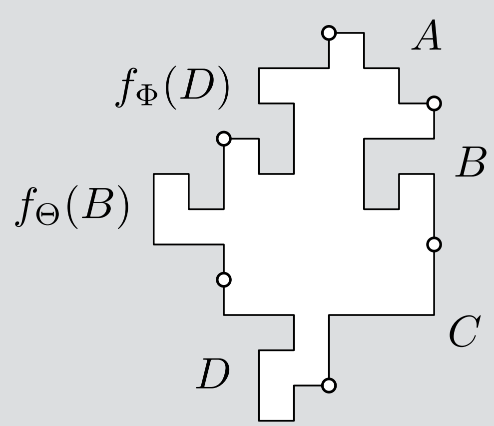

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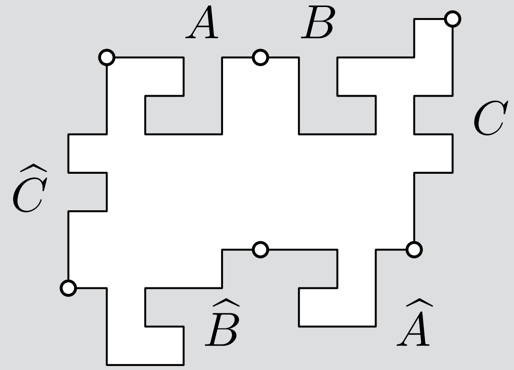

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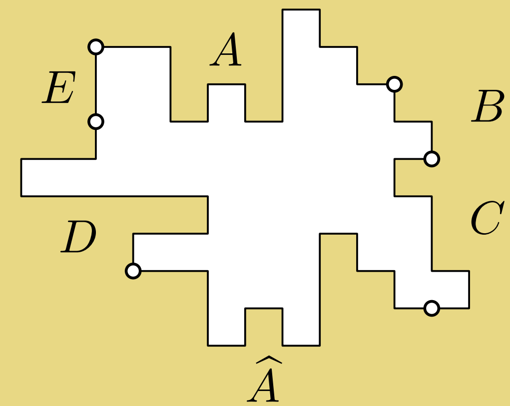

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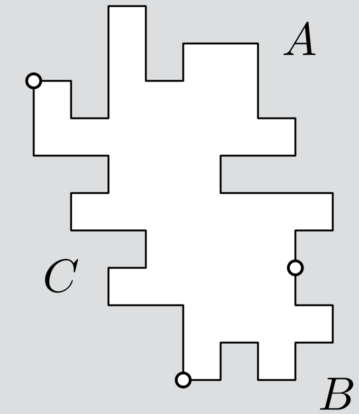

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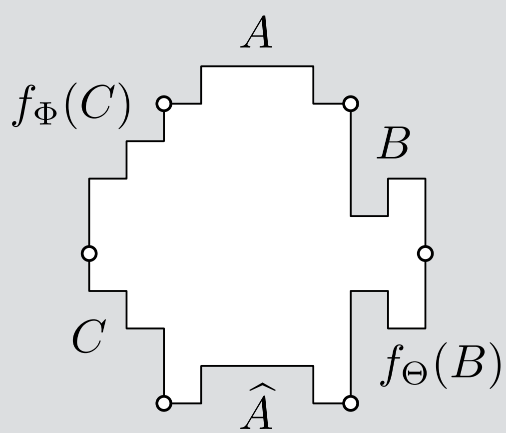
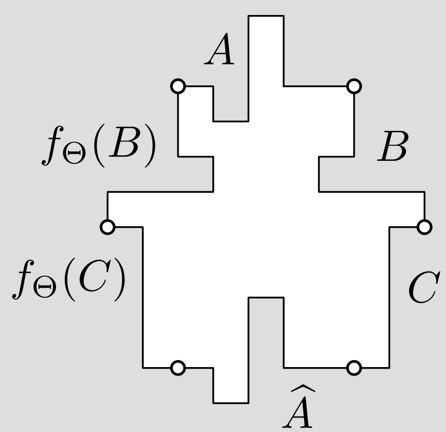
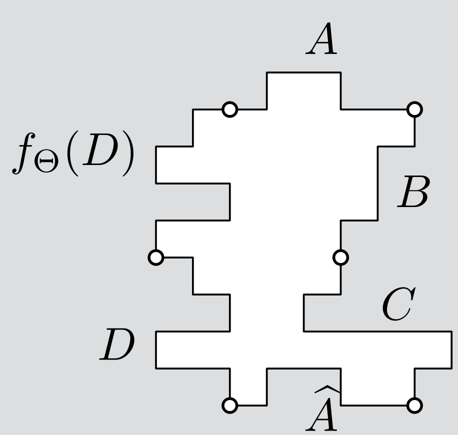
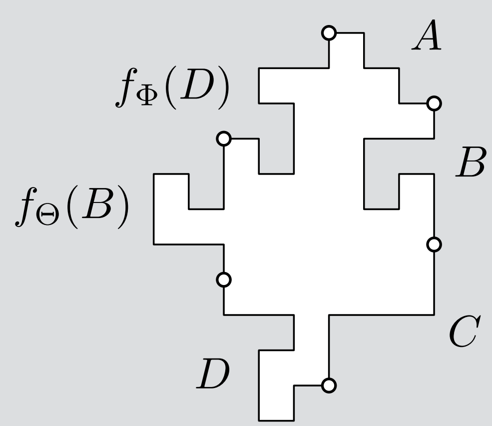






B, C palindromes

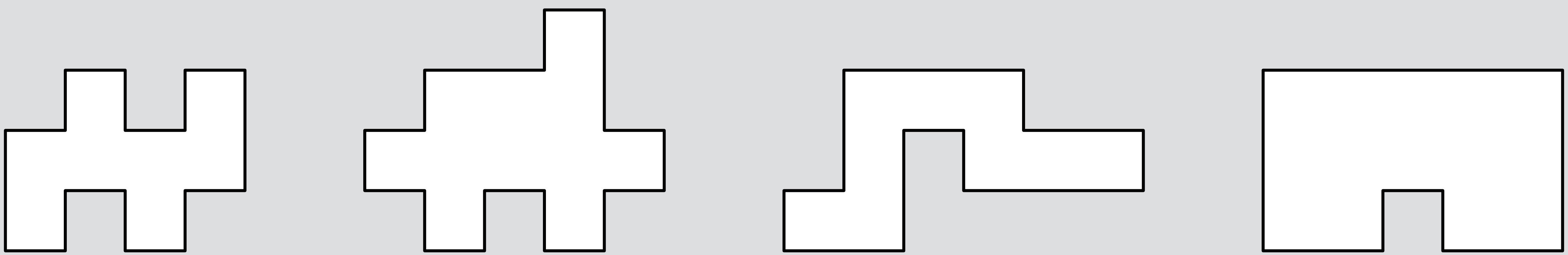


A, C palindromes
$\Theta° - \Phi° = \pm 90°$

Problem

Decide whether a polyomino has an isohedral tiling.

(passes any of 7 criteria)



Yes Yes No No

Prior work

(input polyomino with n sides)

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General case (all 7 criteria):

- [Keating, Vince 1999]: $O(n^{18})$
- Naive checking of criteria: $O(n^6)$
- This work: $O(n*\log^2(n))$

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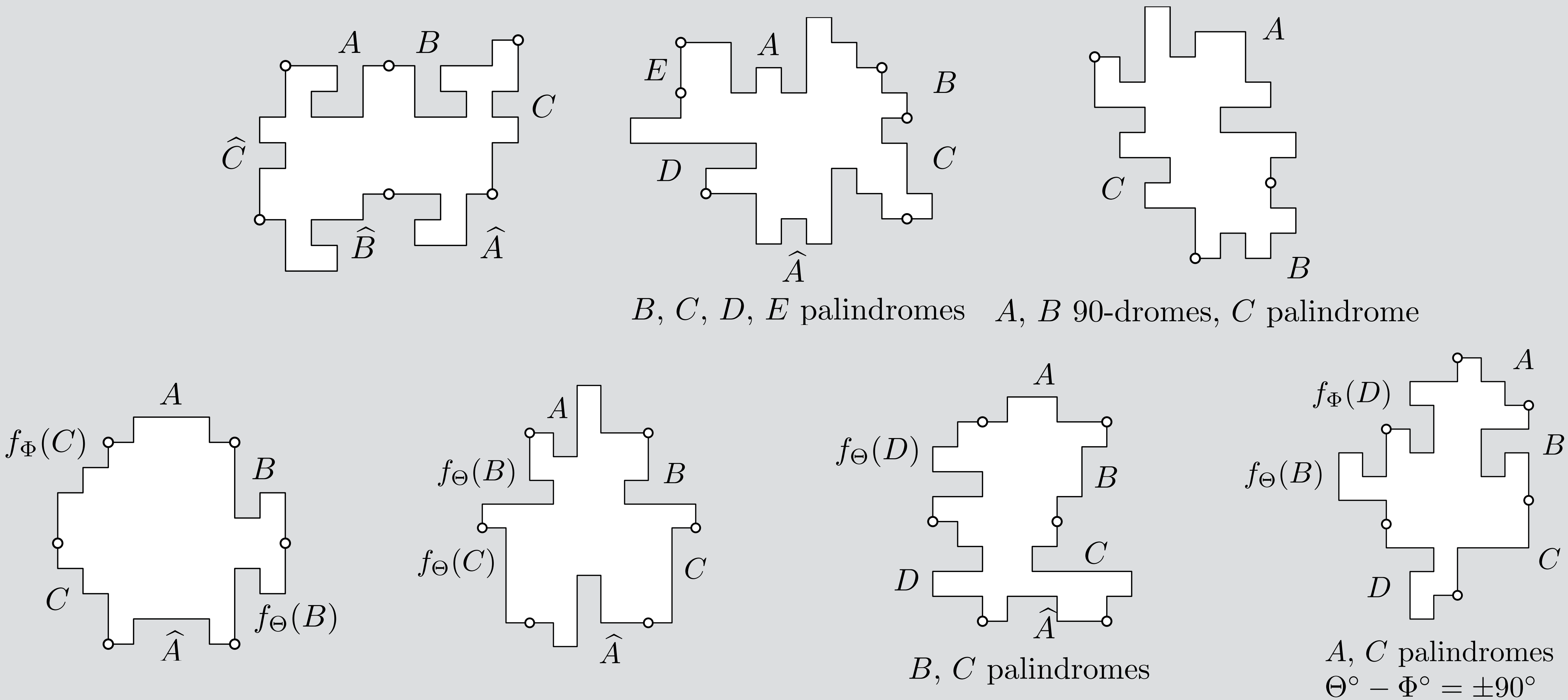
Translation criterion only:

- [Gambini, Vuillon 2007]: $O(n^2)$
- [Provençal 2008]: $O(n*\log^3(n))$
- [Brlek, Provençal, Fédou 2009]: $O(n)$ (special cases)
- [W. 2015]: $O(n)$

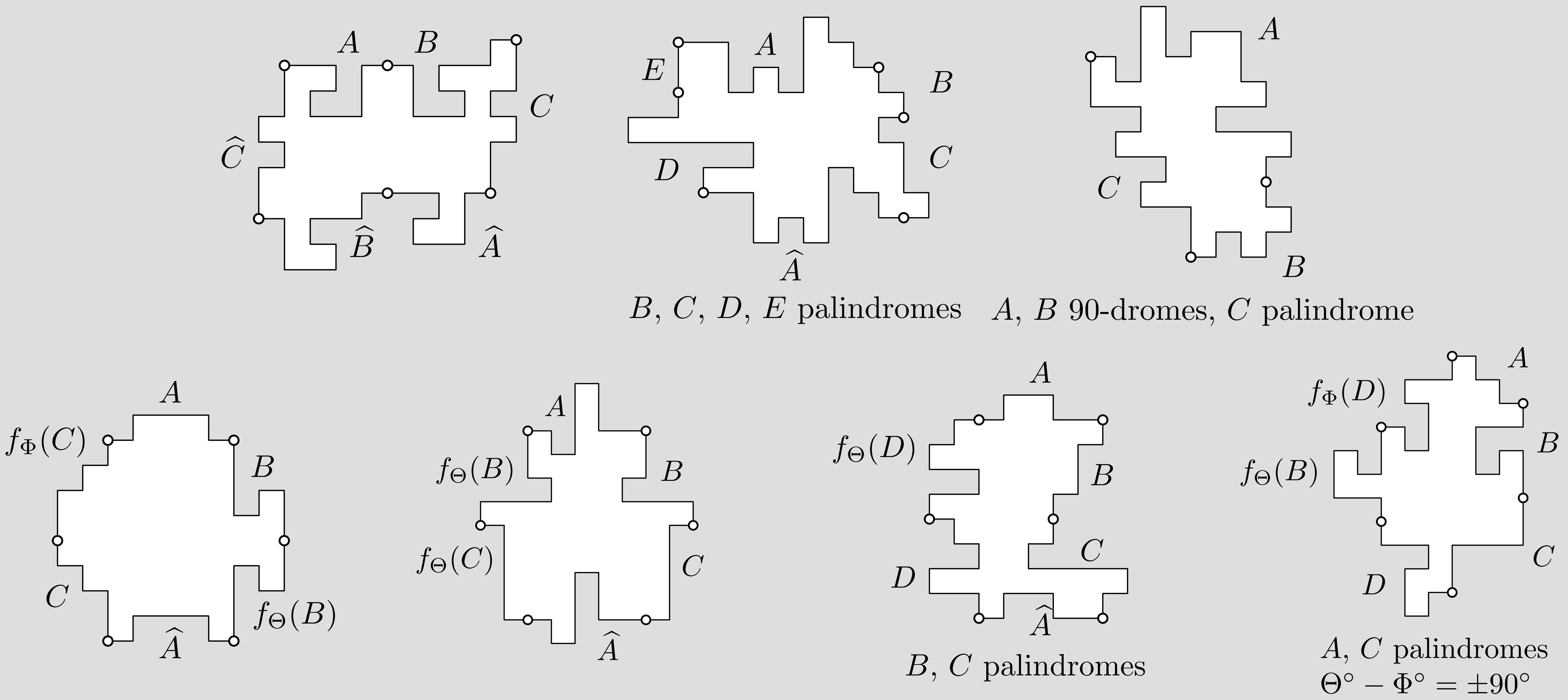
Algorithm

Test the input boundary for each criterion.



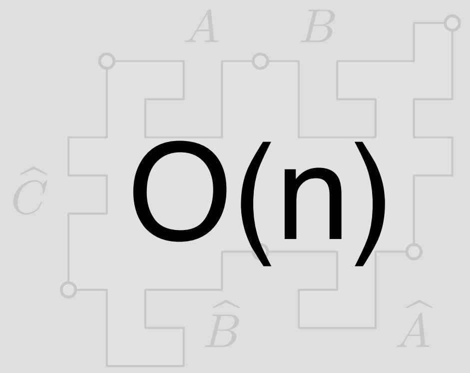
Fast via several technical lemmas on words.

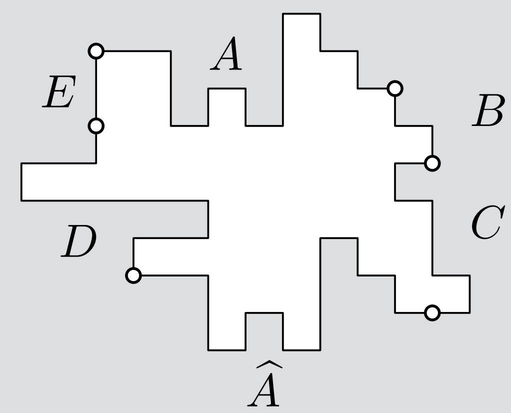
Algorithm running times



Algorithm running times

O(n)





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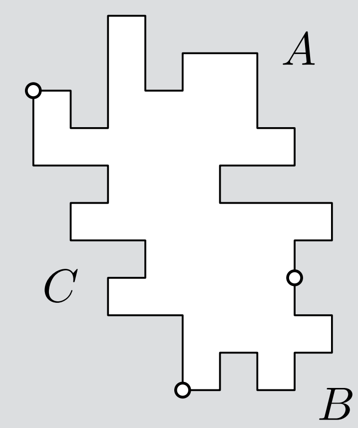

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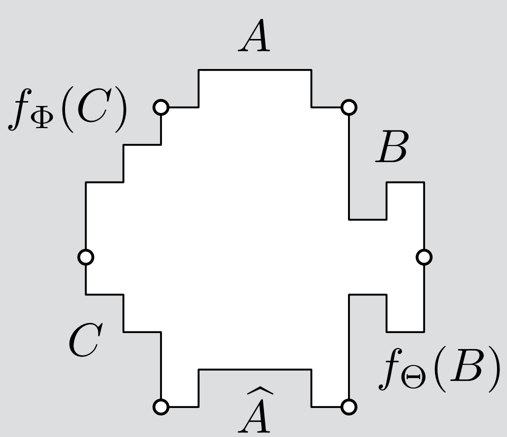
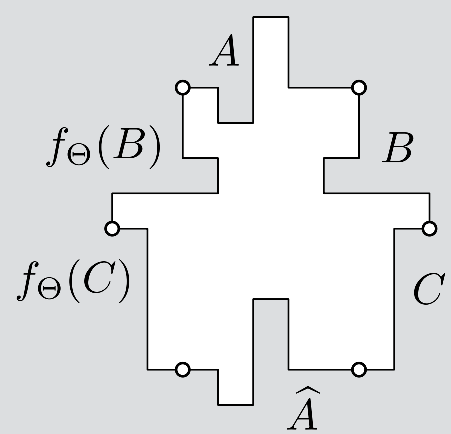
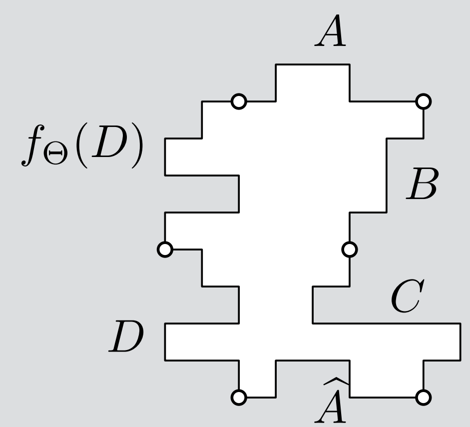
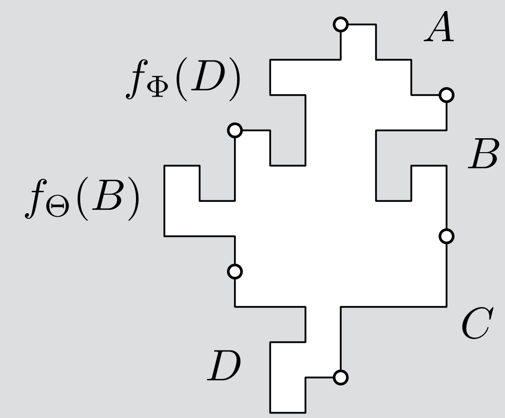






B, C palindromes



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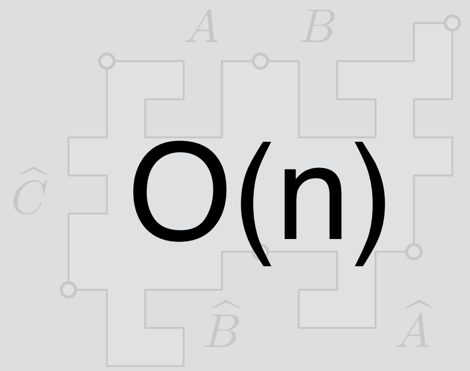
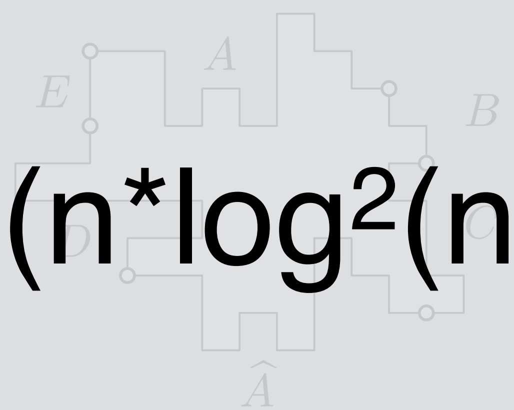
$\Theta^\circ - \Phi^\circ = \pm 90^\circ$

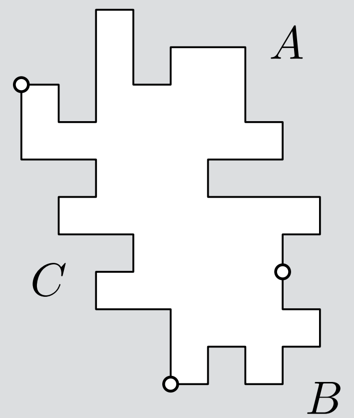
Algorithm running times





B, C, D, E palindromes

O(n)   O(n*log²(n))



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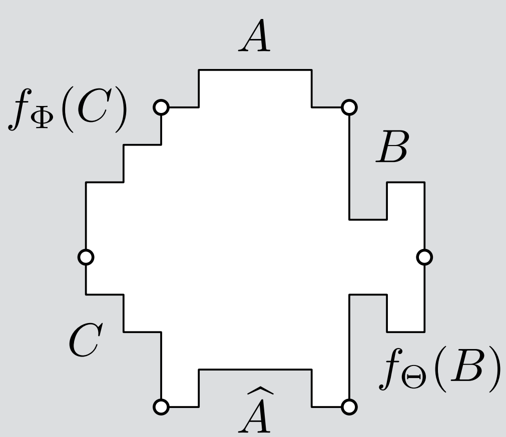
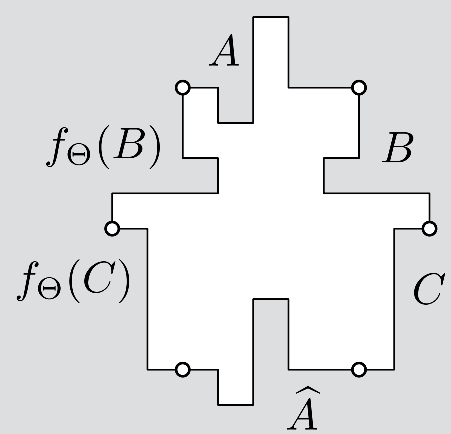
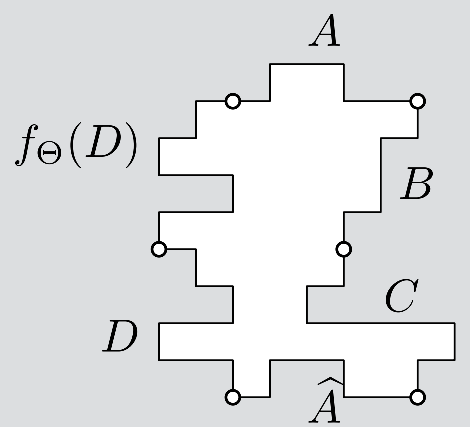
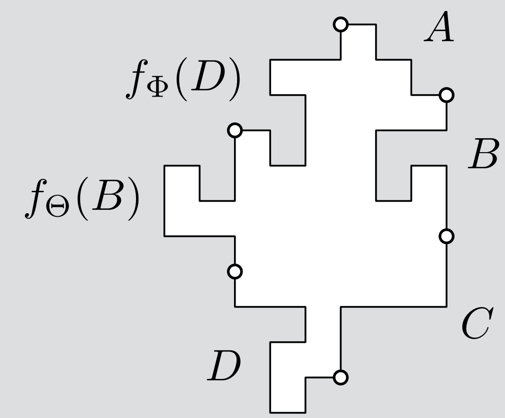






B, C palindromes



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Algorithm running times

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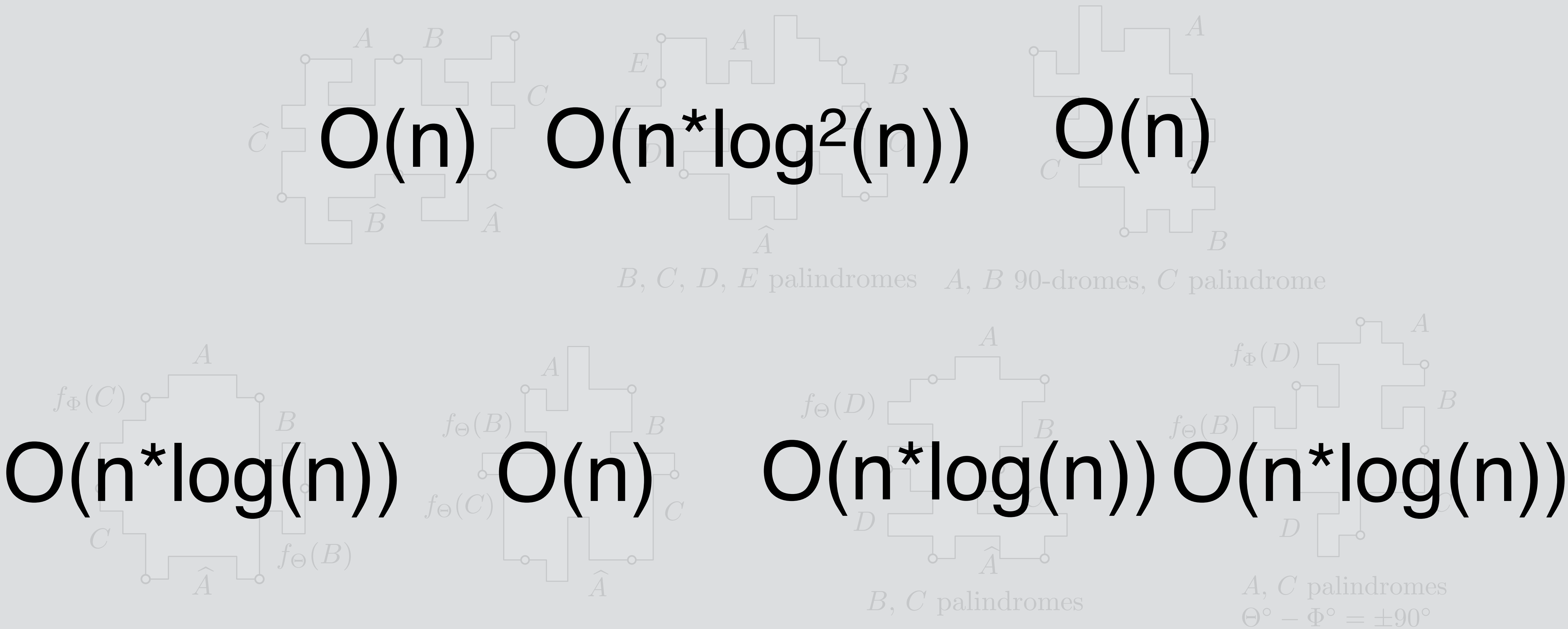
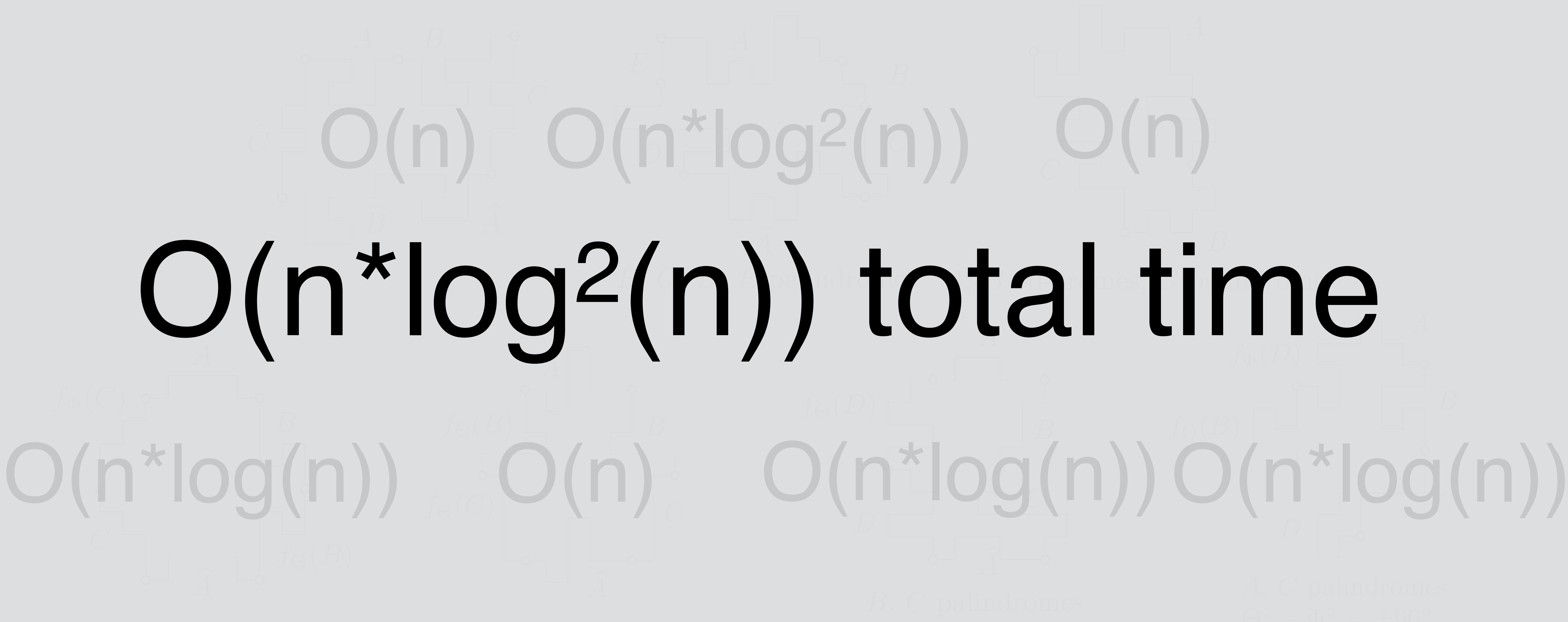


Algorithm running times



$O(n*\log^2(n))$ total time

Open Problems

Theorem: $O(n*\log^2(n))$-time algorithm for deciding if a polyomino tiles the plane isohedrally.

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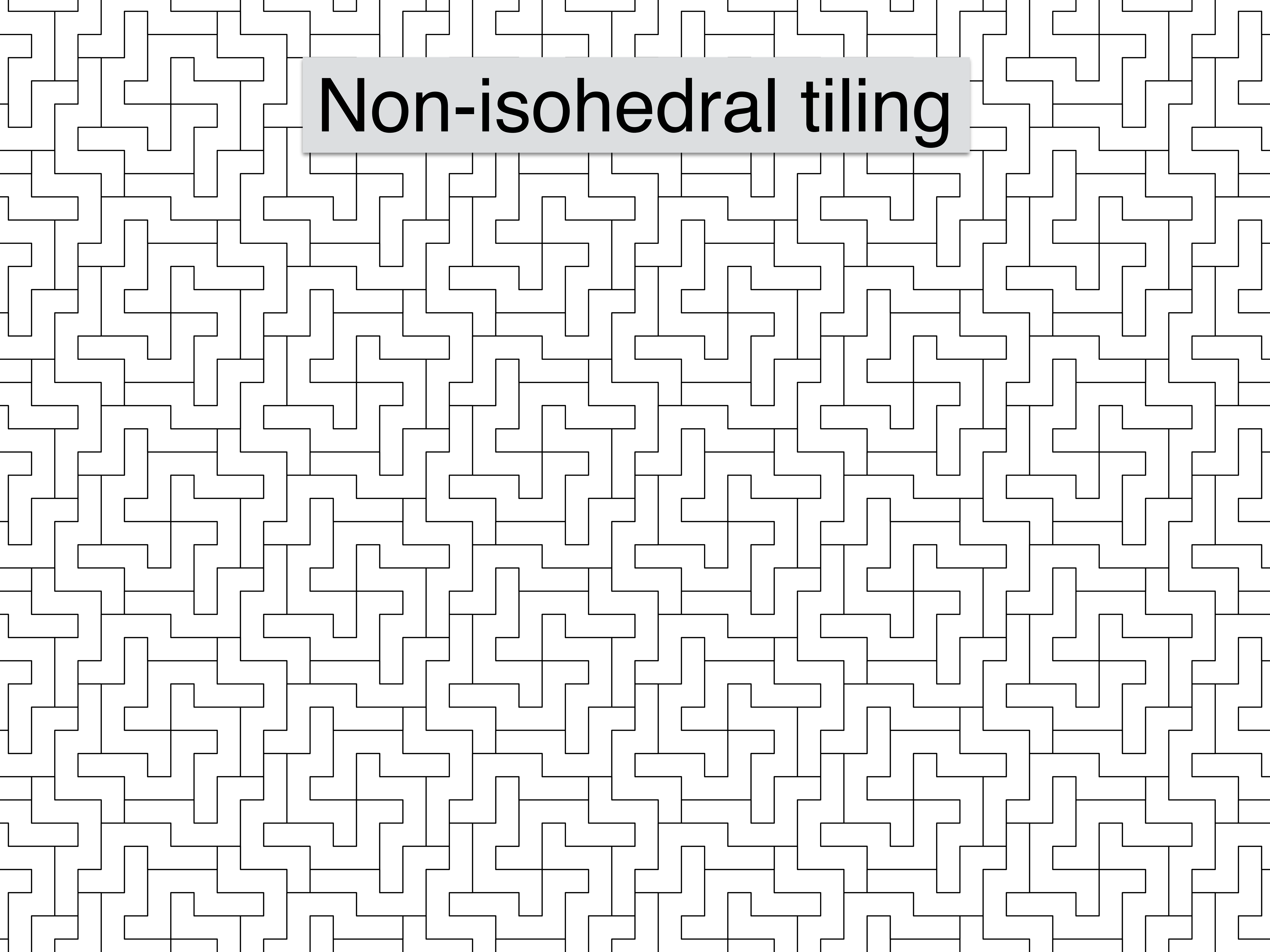
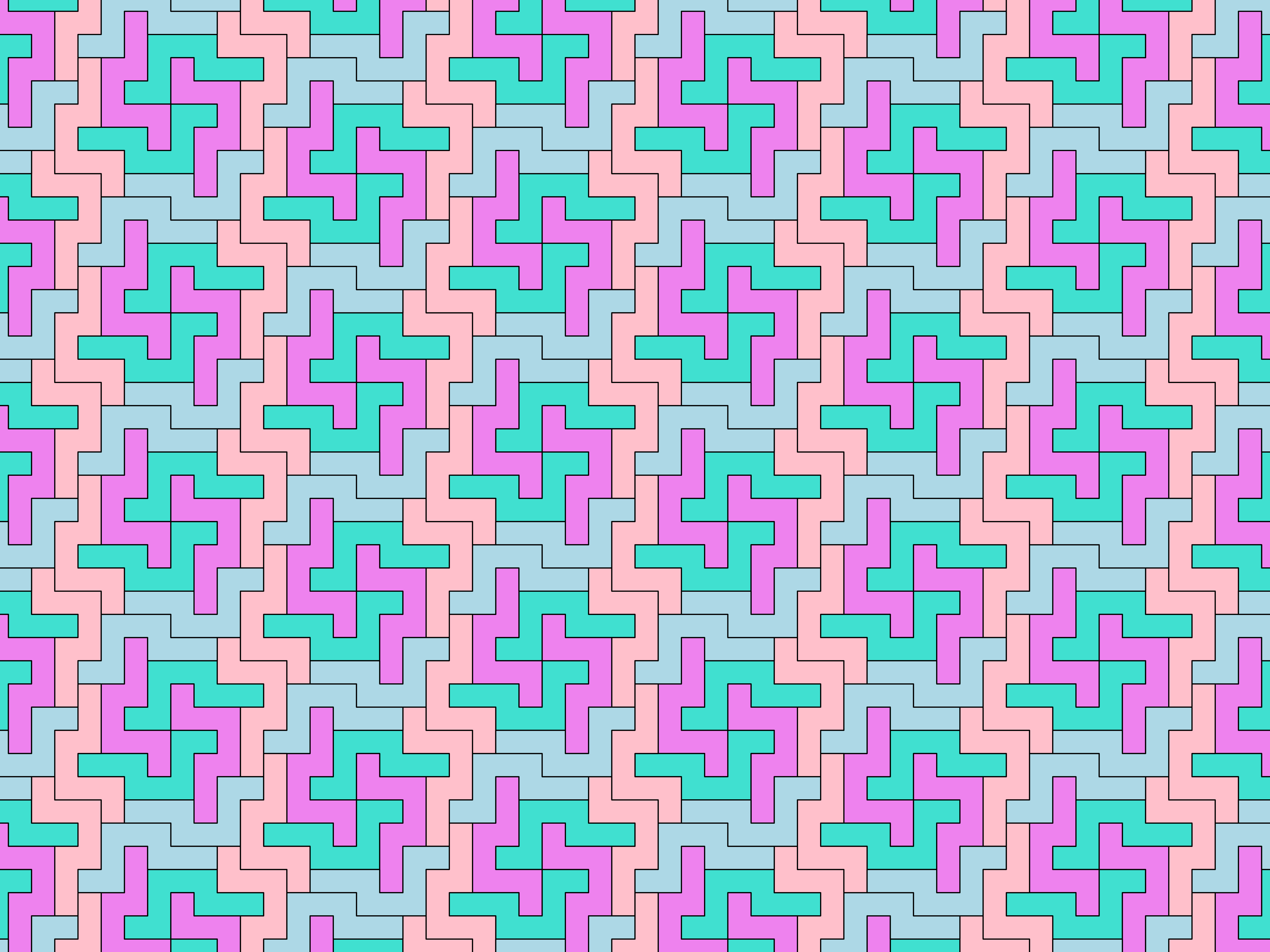
Enumeration of tilings in $O(n*\log^2(n) + t)$ time?

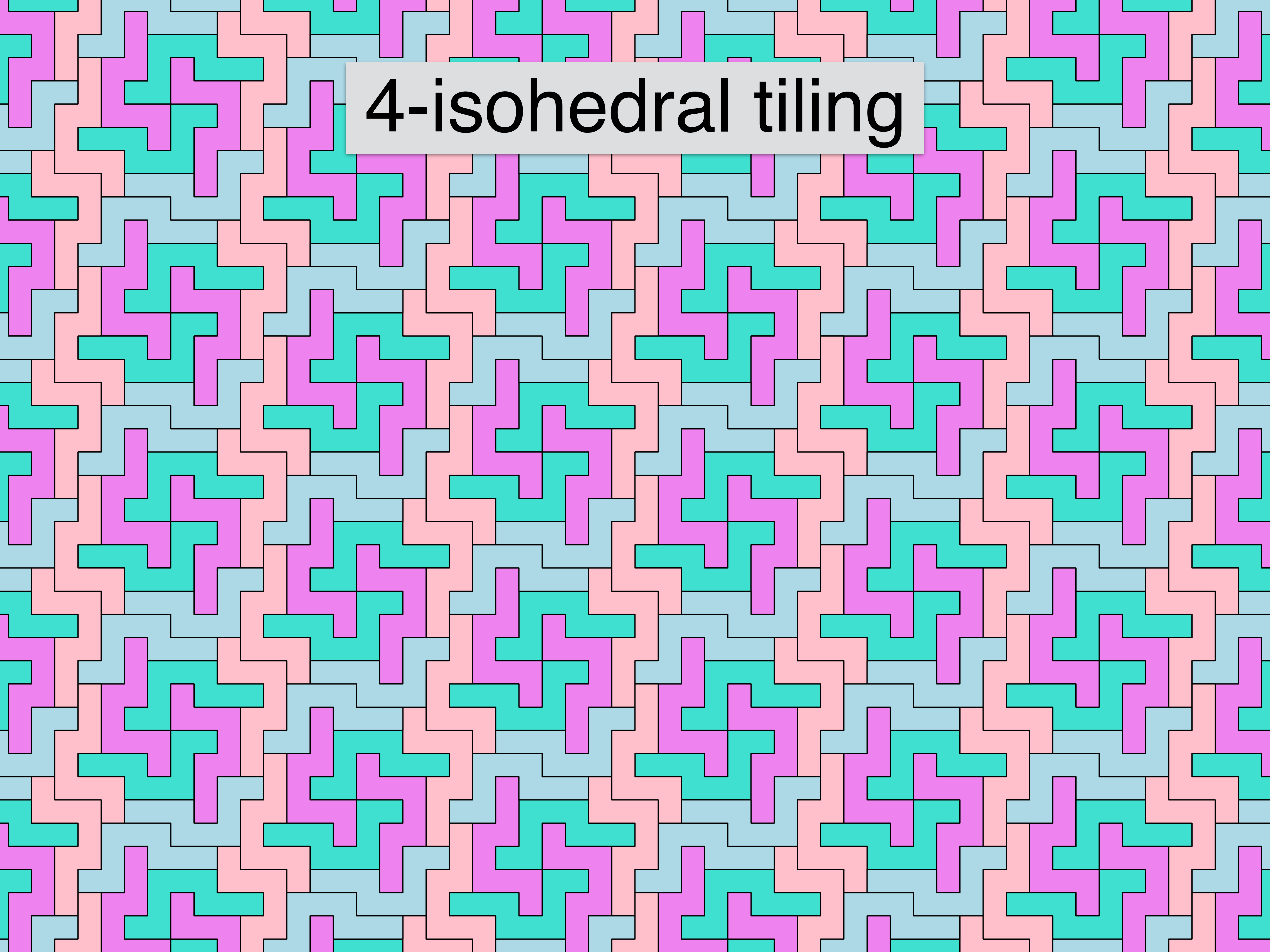
Extend inputs to polygons?

Non-isohedral tiling



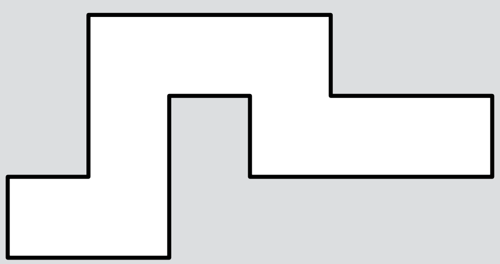


4-isohedral tiling



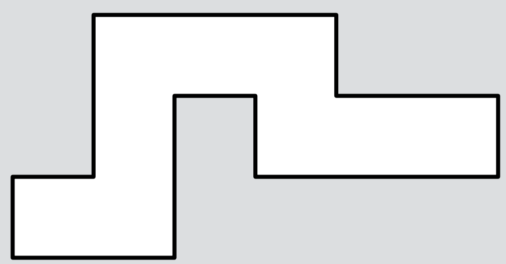
Problem

Decide whether a polyomino has a ≤k-isohedral tiling.



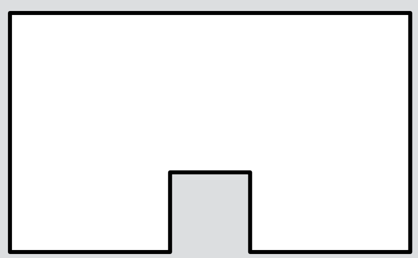
$k = 5$

Yes



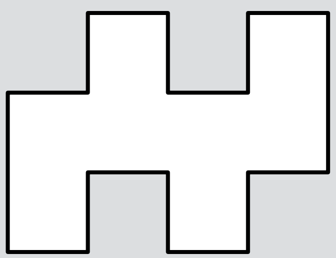
$k = 3$

No



$k = 3$

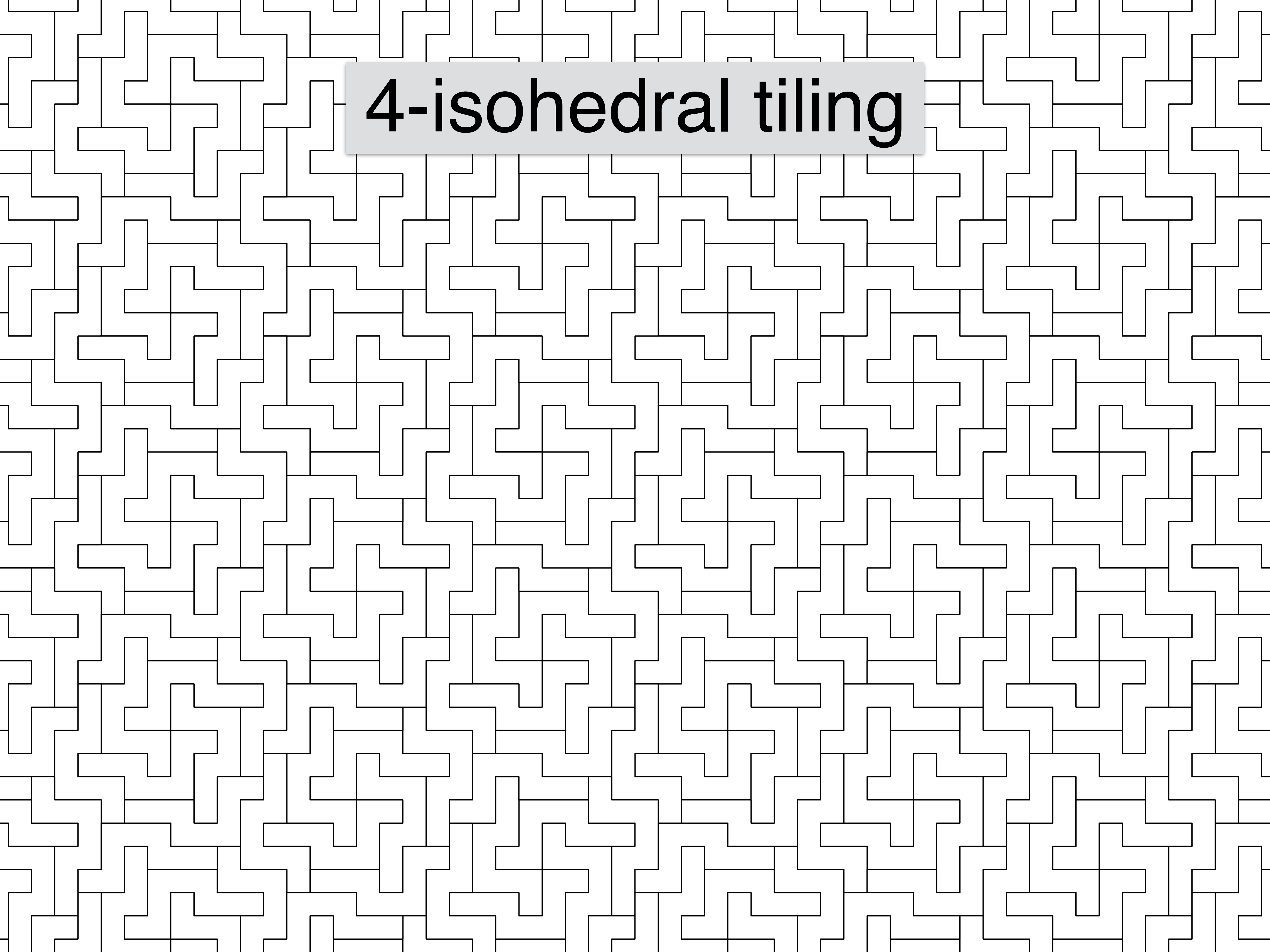
No

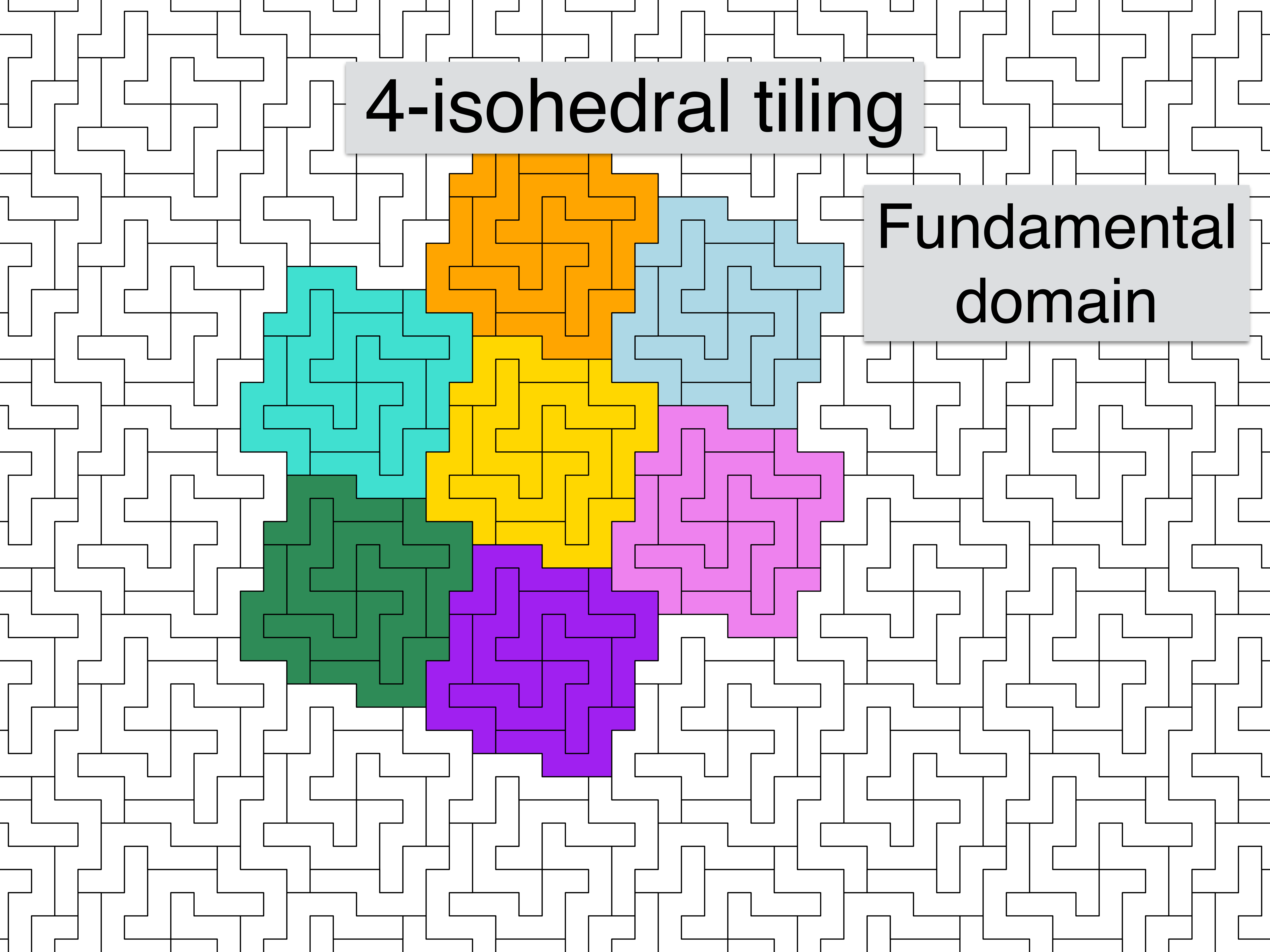


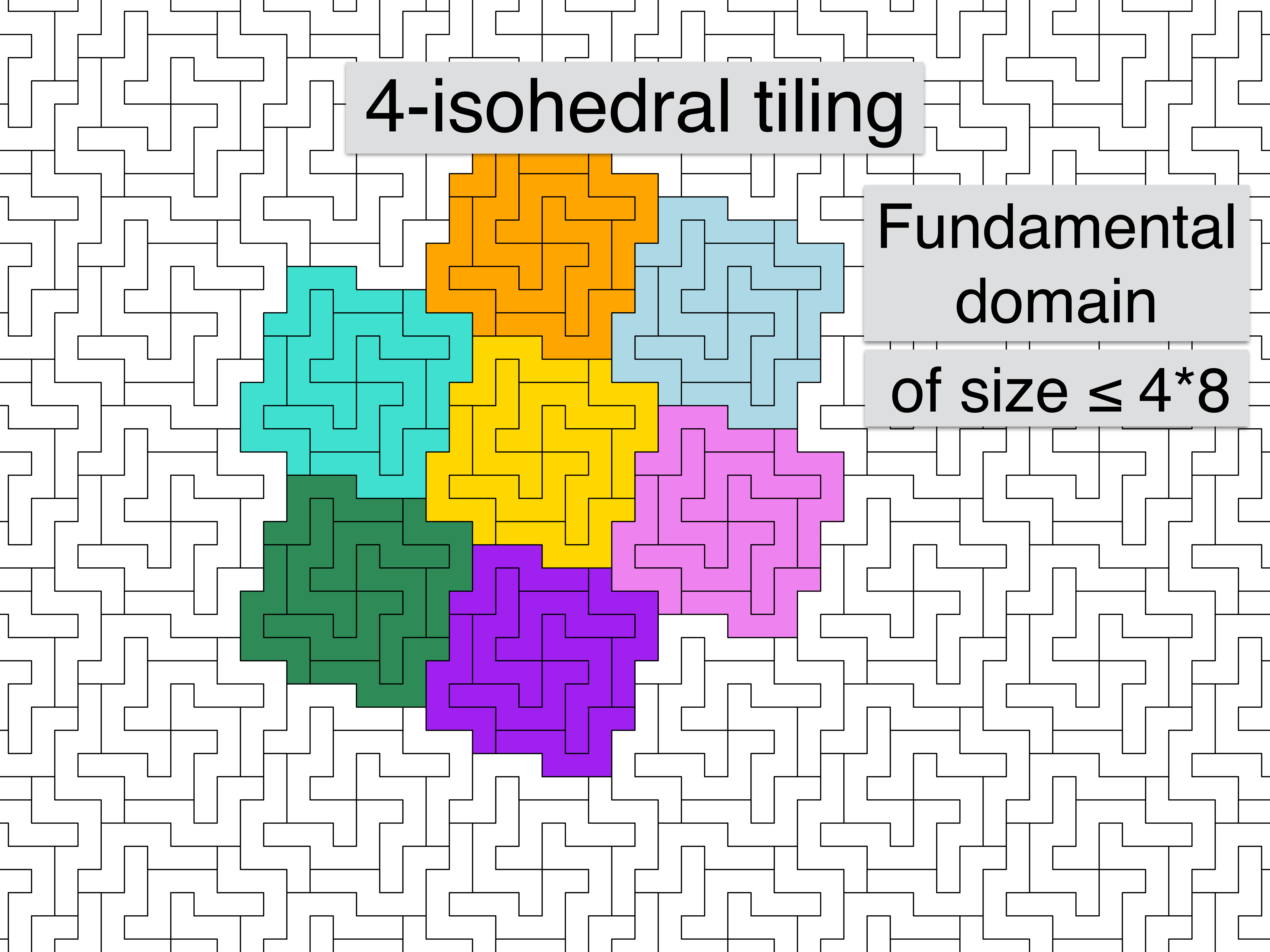
$k = 1$

Yes

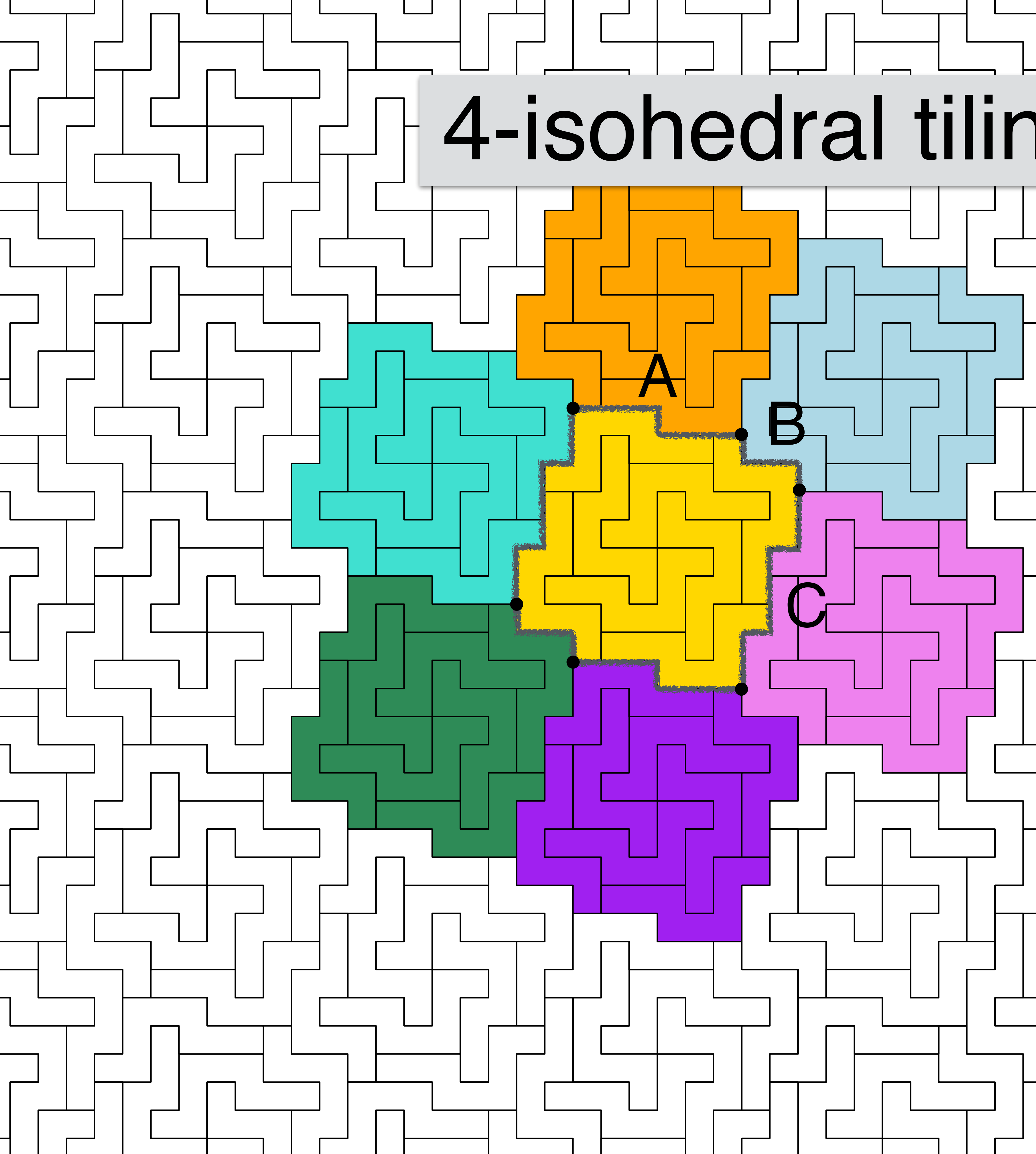
4-isohedral tiling




4-isohedral tiling
Fundamental domain


4-isohedral tiling
Fundamental domain
of size ≤ 4*8

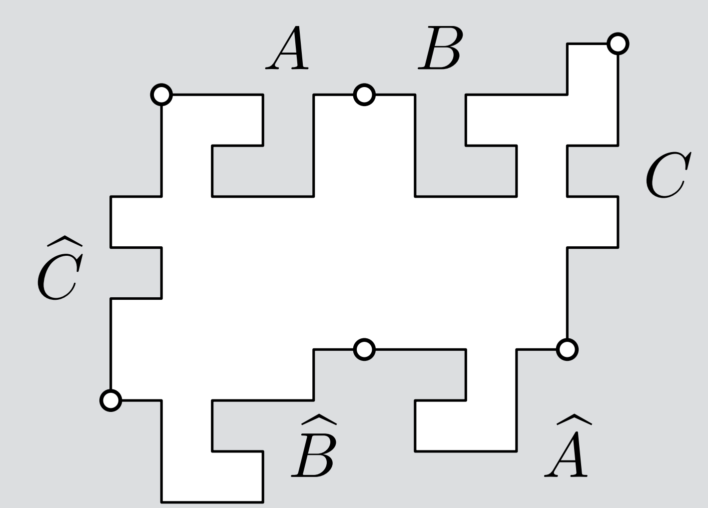
4-isohedral tiling



Fundamental domain

of size ≤ 4*8

satisfies



translation criterion

Problem

Decide whether a polyomino

has a ≤k-isohedral tiling.

Algorithm

Check every fundamental domain

of size ≤ k*8 for translation criterion.

Problem

Decide whether a polyomino has a ≤k-isohedral tiling.

Algorithm

Check every fundamental domain of size ≤ k*8 for translation criterion.

$n^{O(k)}$-time

Is there an $f(k)n^{O(1)}$-time algorithm?

50-year-old Problem

Decide whether a polyomino

has a ~~≤k-isohedral~~ tiling.

Is there an ~~$f(k)n^{O(1)}$-time~~ algorithm?

A Quasilinear-Time Algorithm for Tiling the Plane Isohedrally with a Polyomino

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