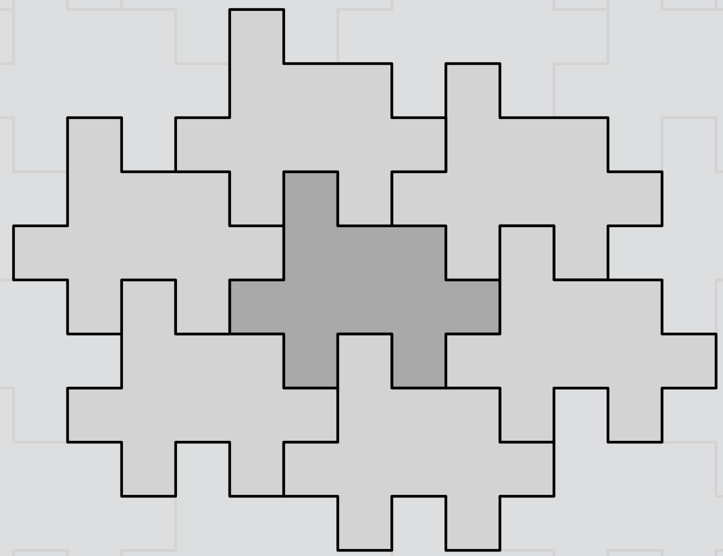
Tiling Isohedrally with a Polyomino

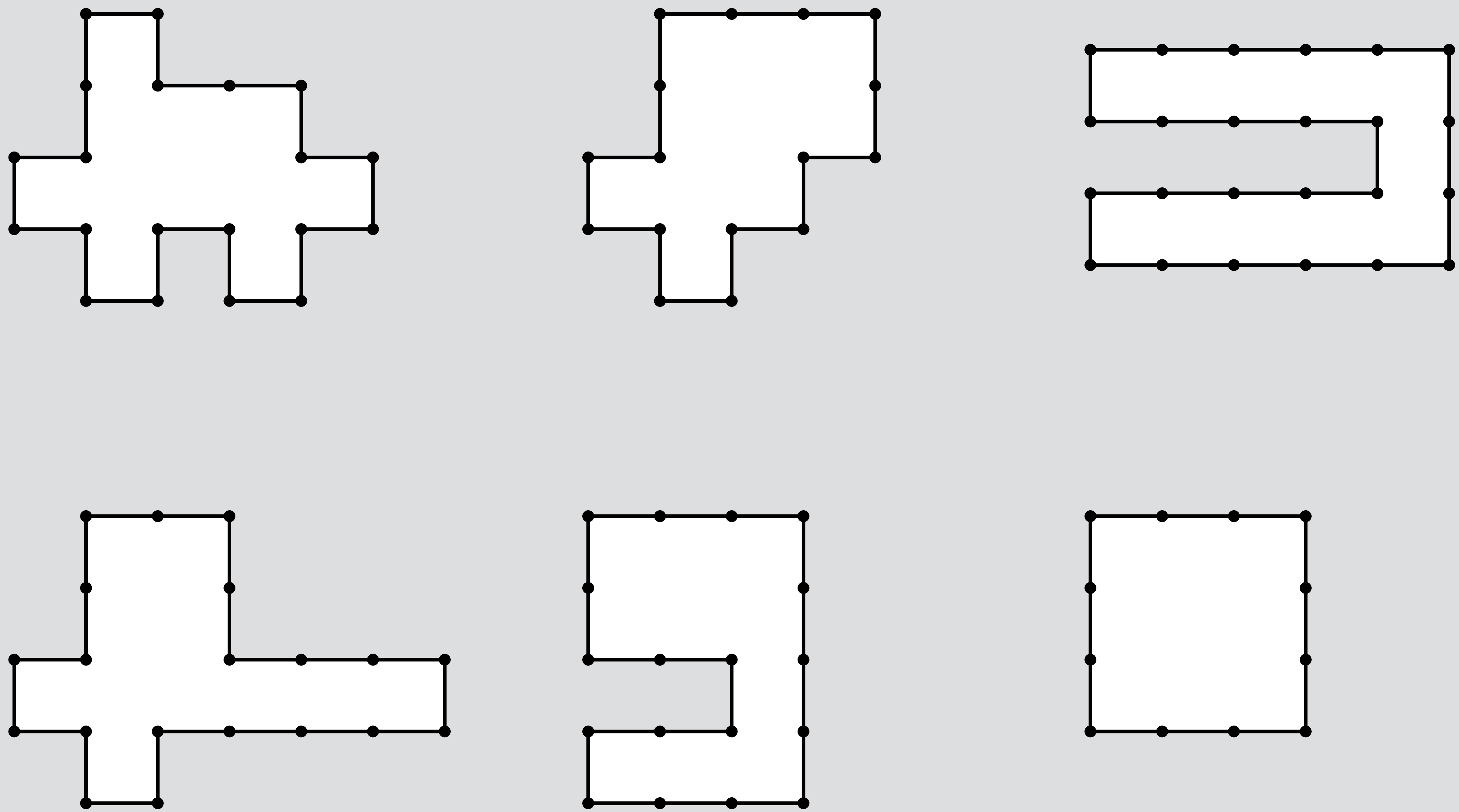


Andrew Winslow
(joint work with Stefan Langerman)

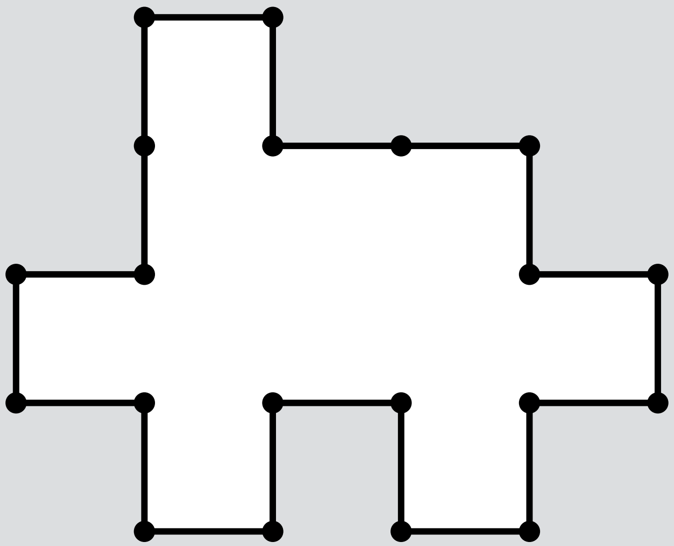
ULB UNIVERSITE LIBRE DE BRUXELLES

Polyominoes

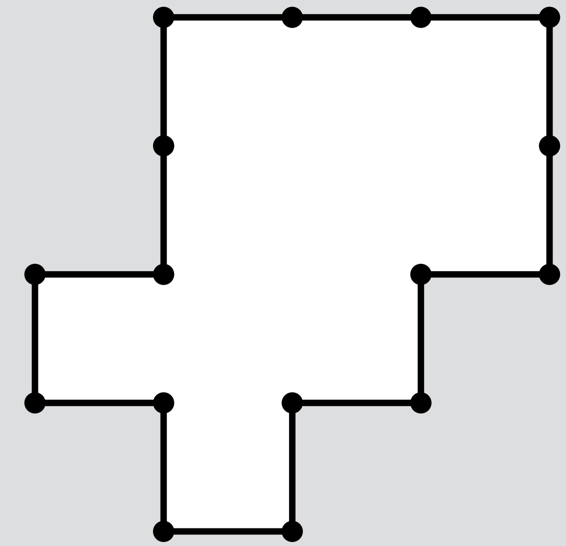
Rectilinear simple polygons with unit edge lengths



Boundary words

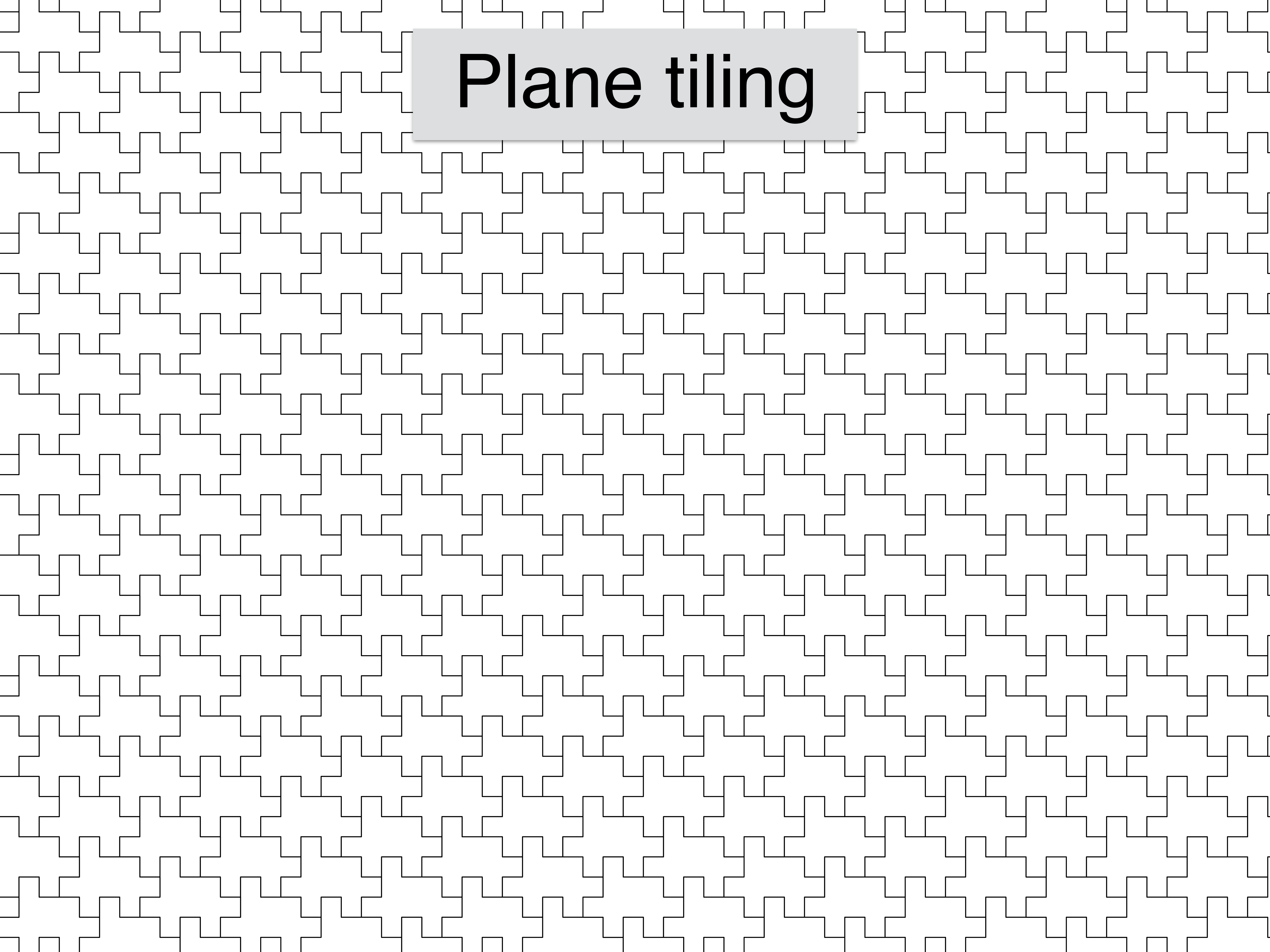


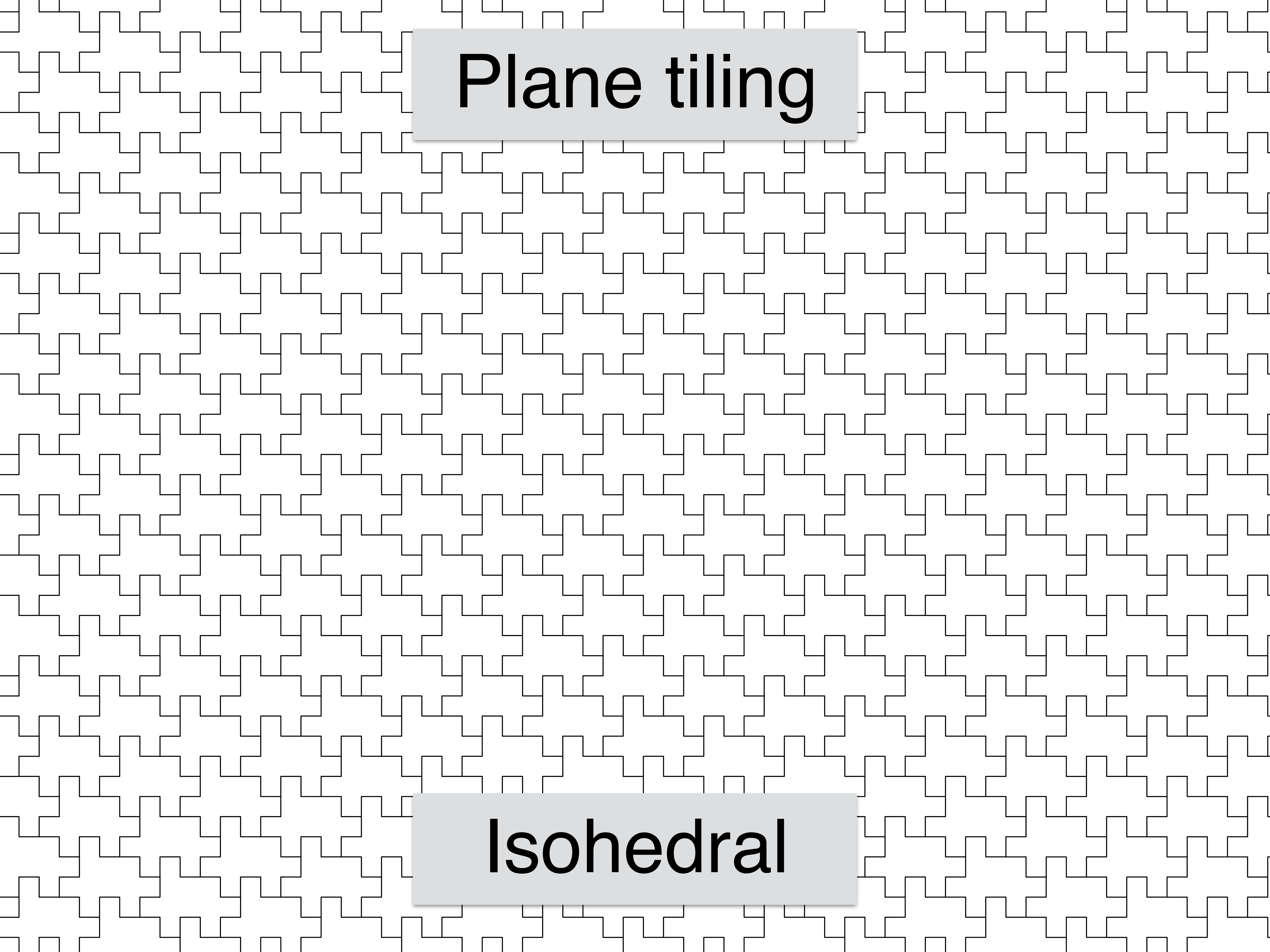
$uru^2rdr^2drd(ldlu)^2l$

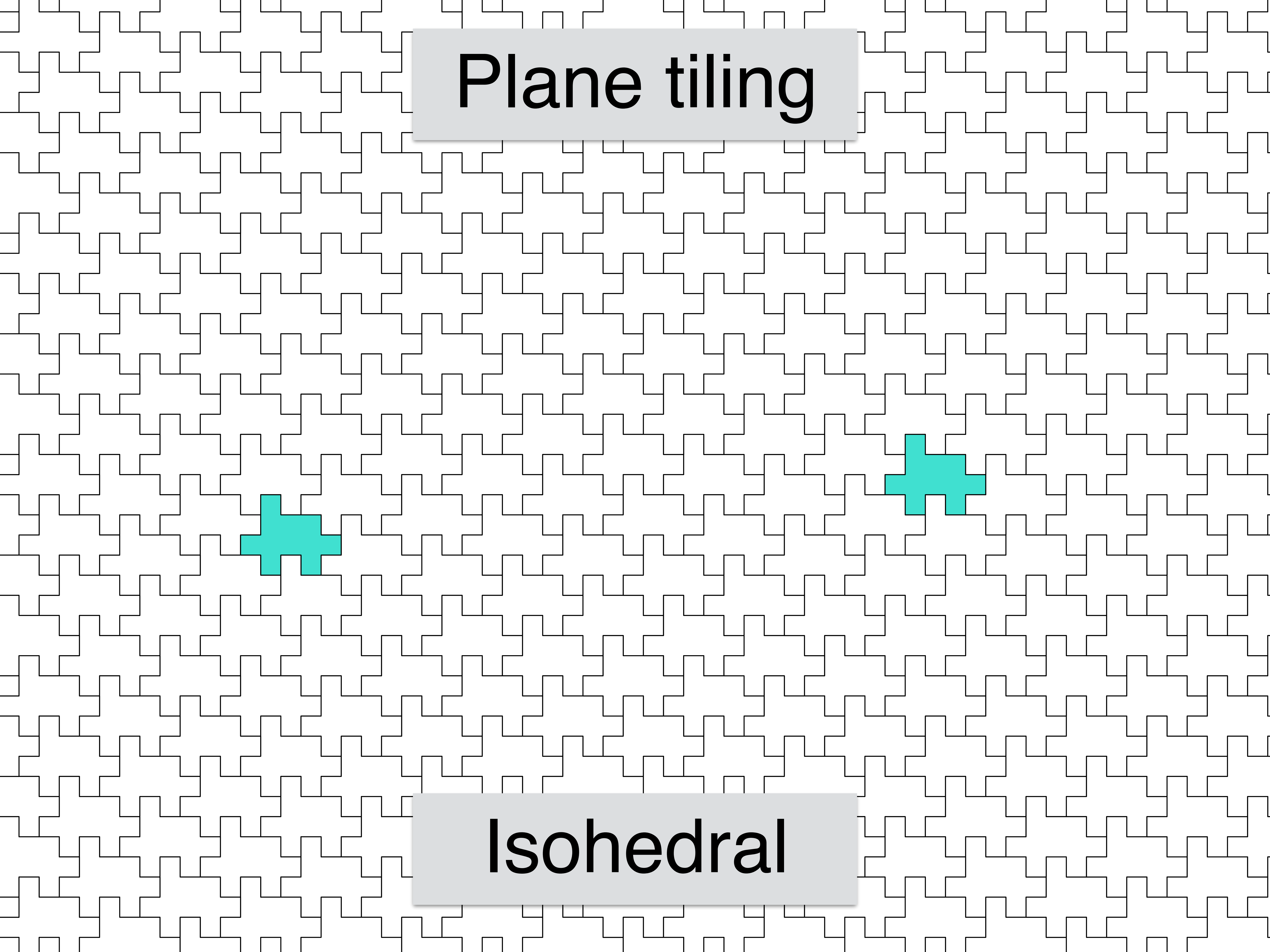


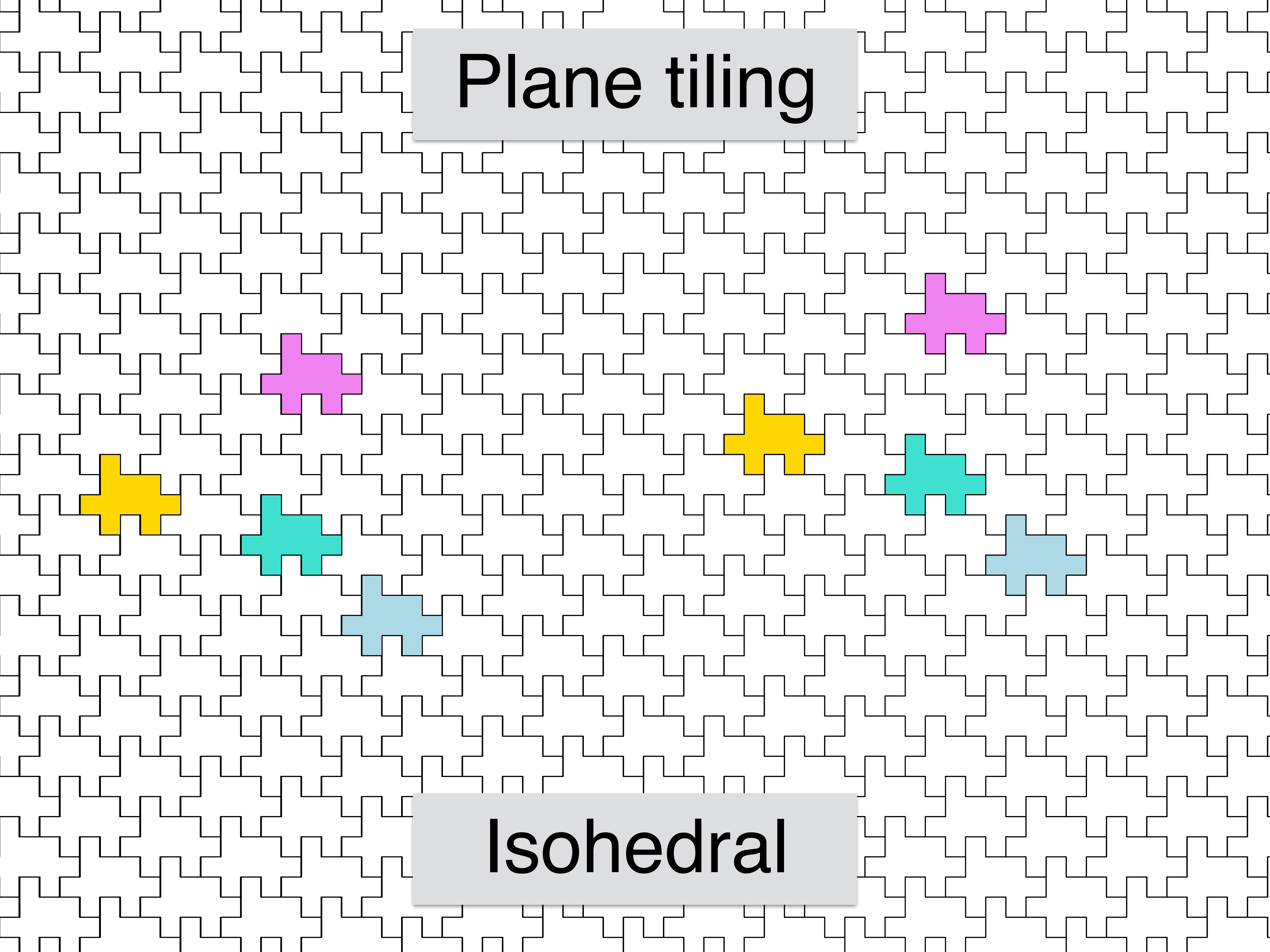
$d(dl)^3uluru^2r^3$

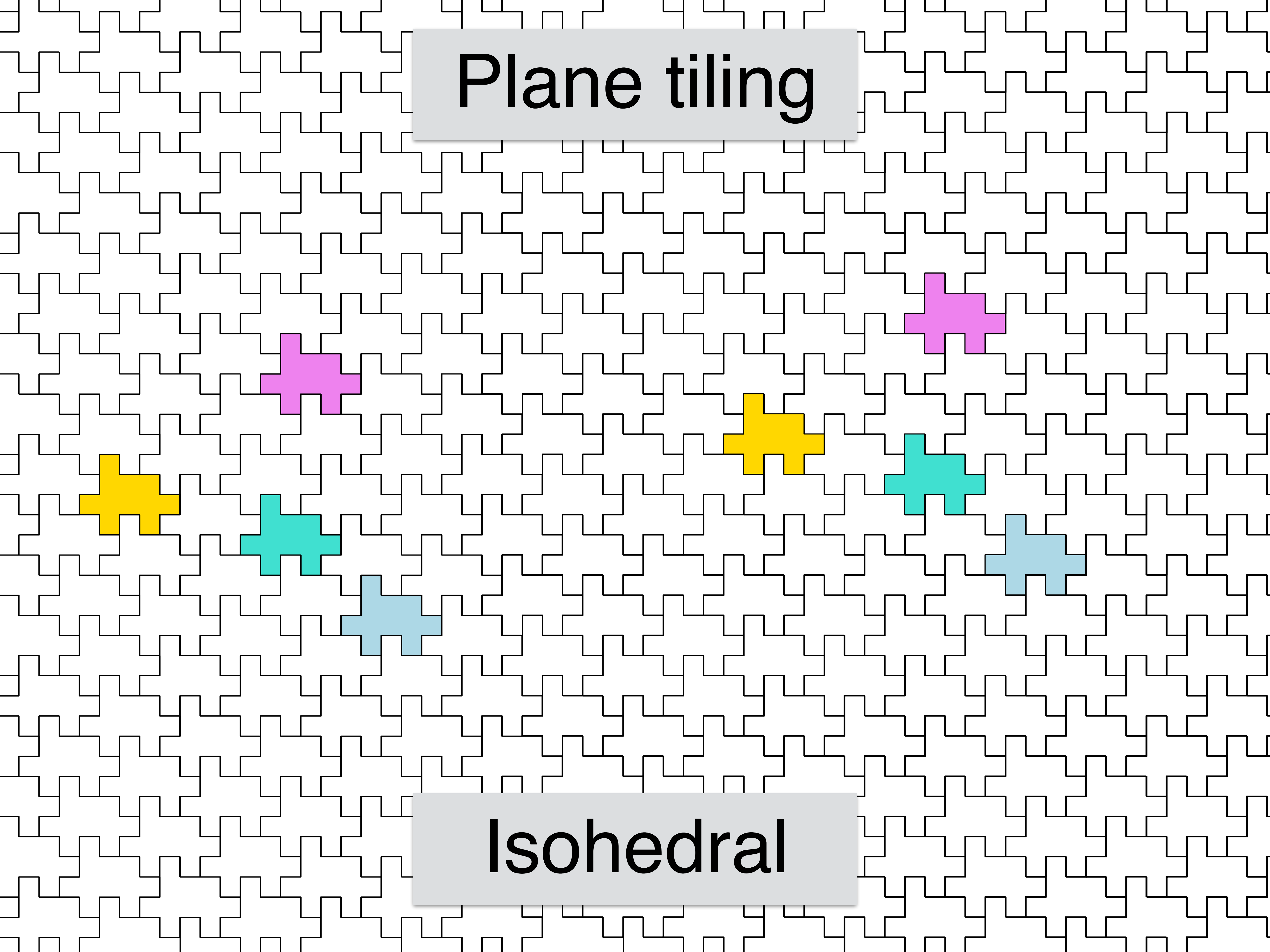
Plane tiling

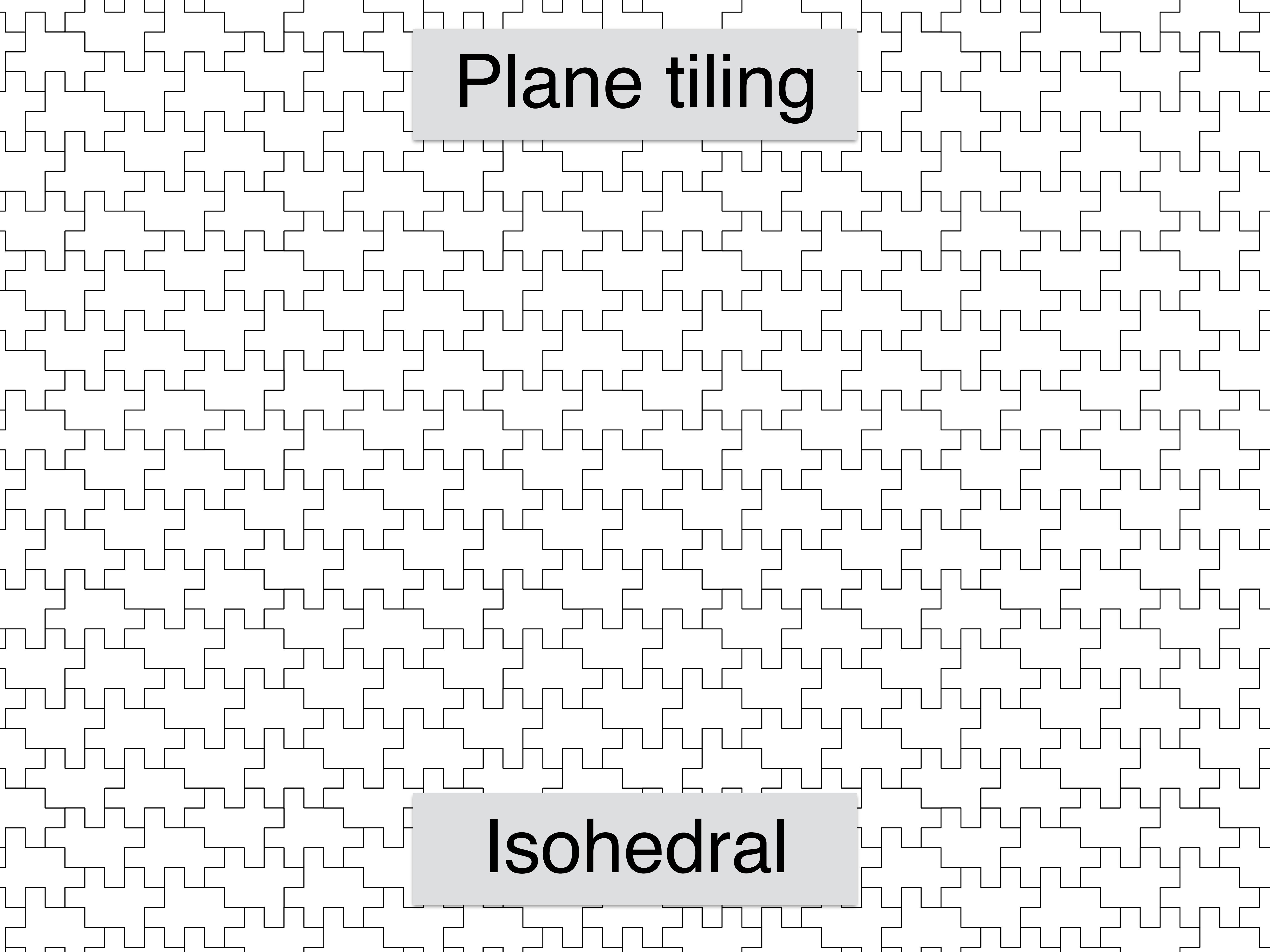



Plane tiling
Isohedral

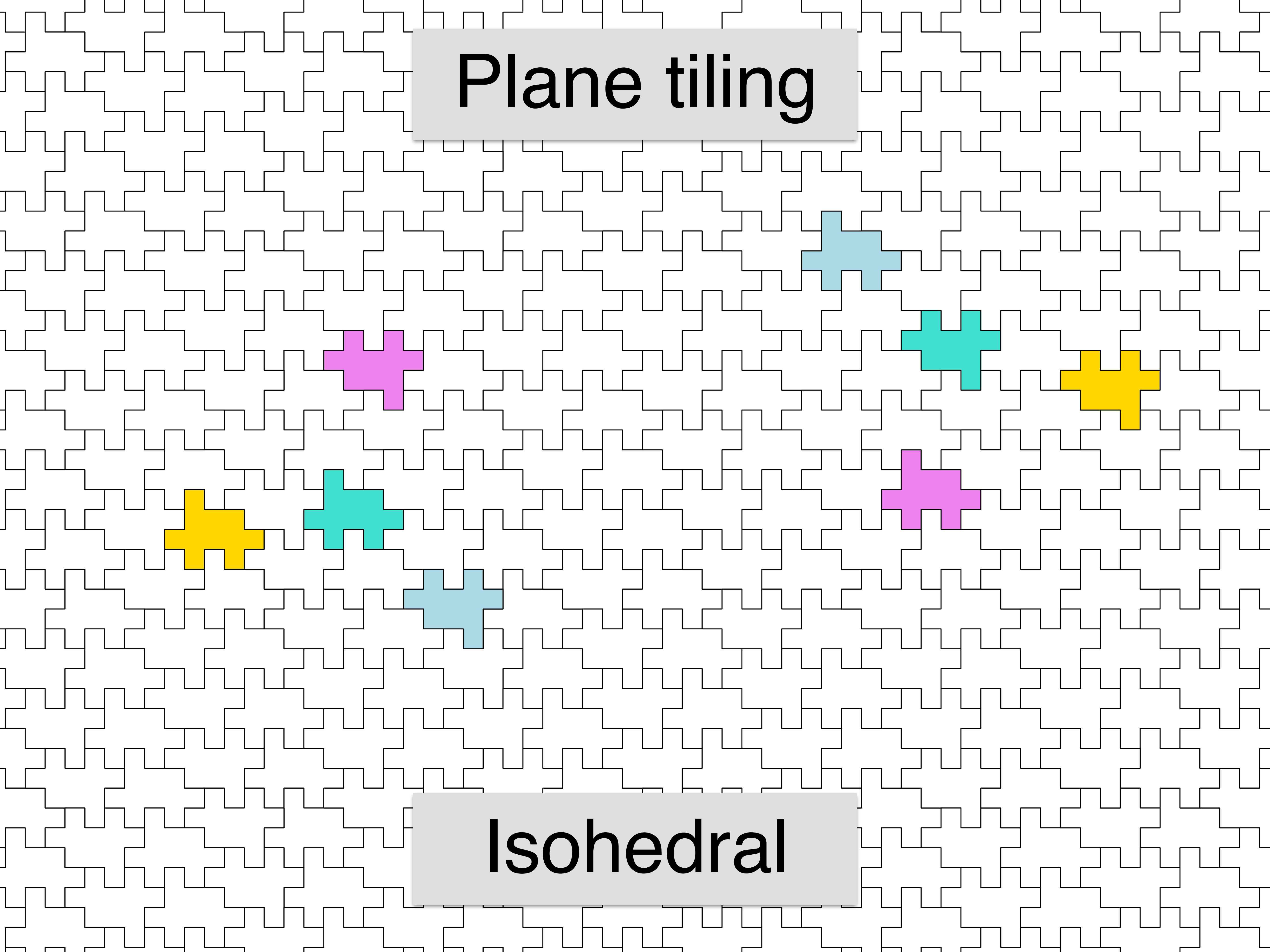

Plane tiling
Isohedral


Plane tiling
Isohedral


Plane tiling
Isohedral

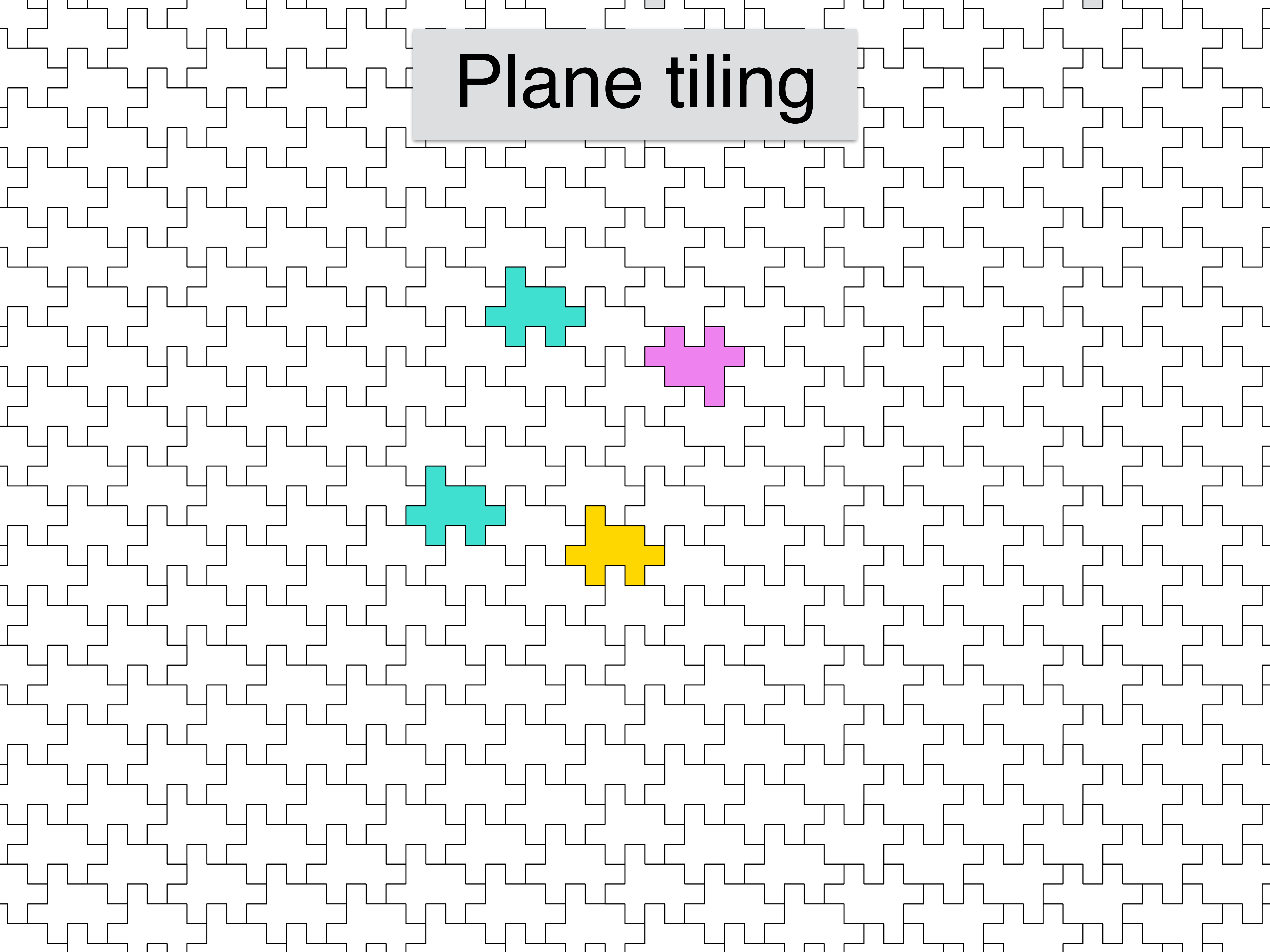

Plane tiling
Isohedral

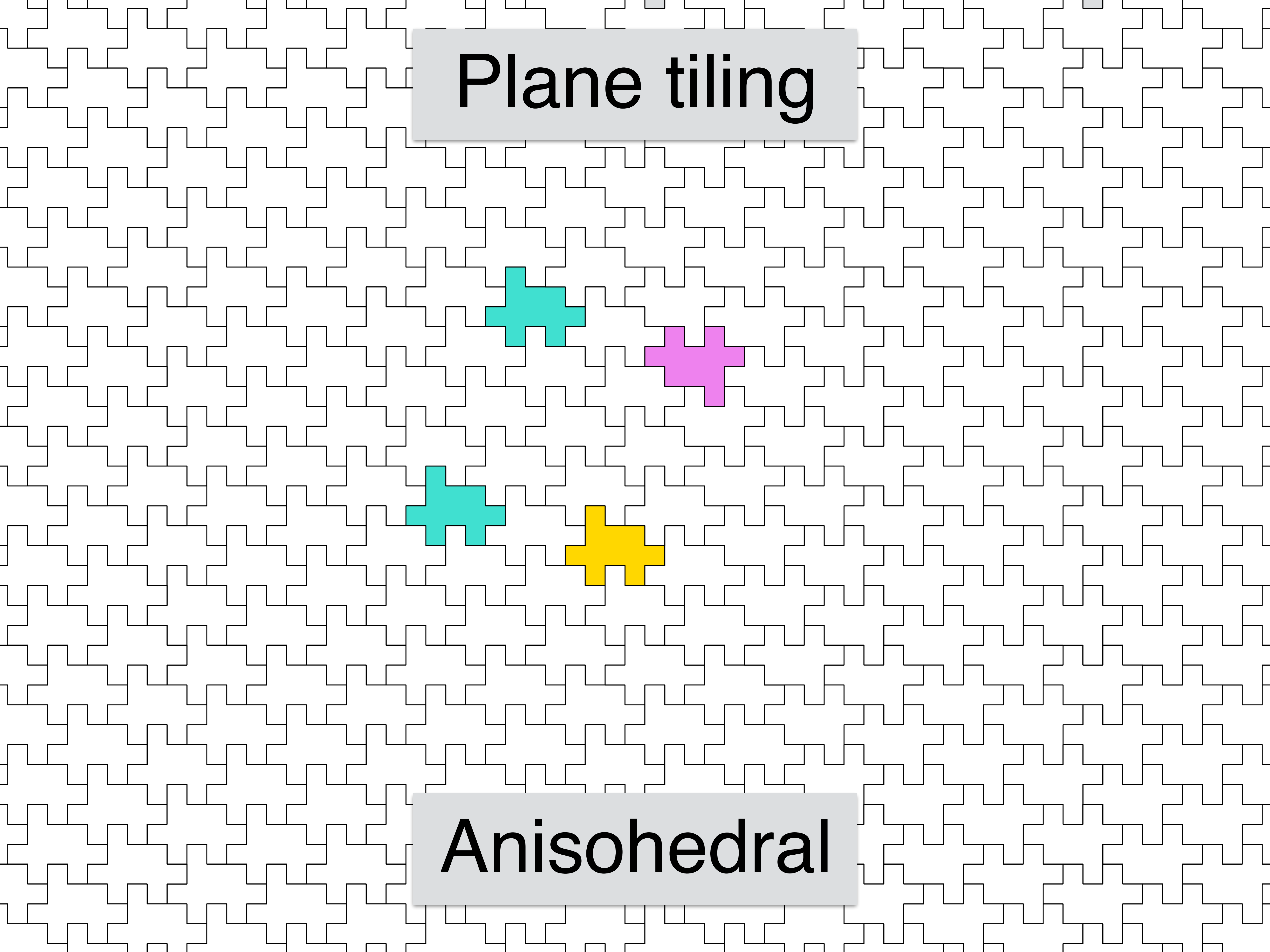
Plane tiling



Isohedral

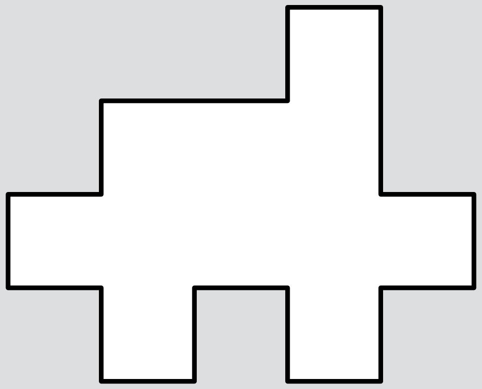
Plane tiling




Plane tiling
Anisohedral

A tiling is either isohedral or anisohedral.

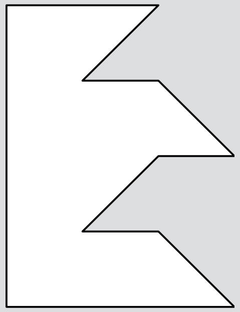
A shape that admits an isohedral tiling is <u>isohedral</u>.



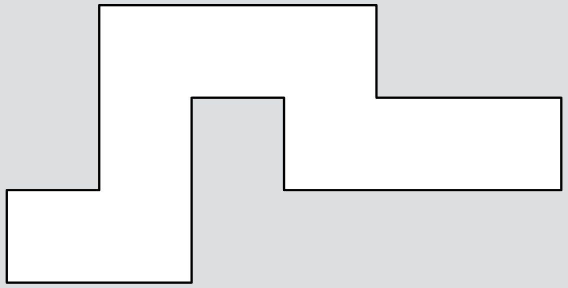
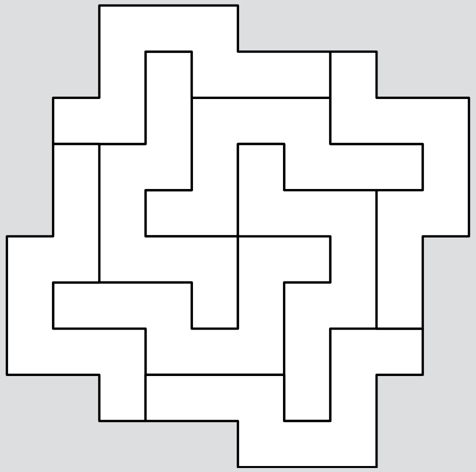
A shape that admits a tiling, but no isohedral tiling is <u>anisohedral</u>.

Are there anisohedral shapes?

1902: Hilbert thinks no.
Premise of 18th of his famous 23 problems.

1935: Heesch proved yes. 

1968: Kershner proved yes for convex shapes.

[Rhoads 2005]:  

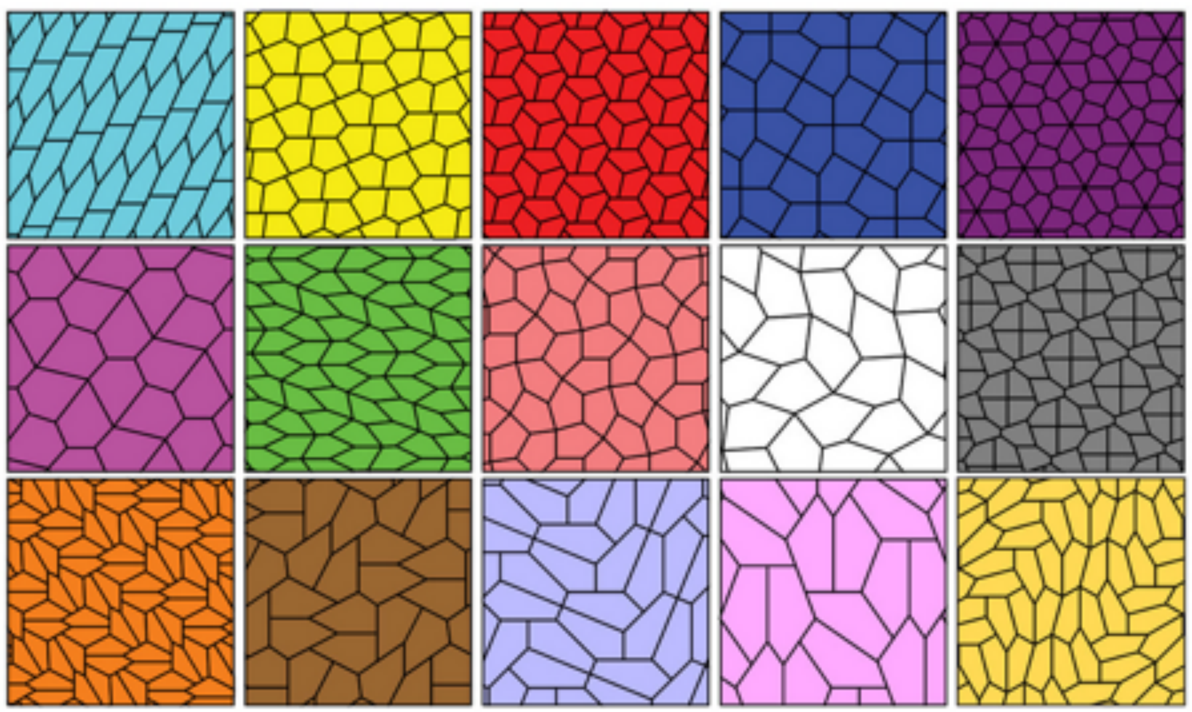
A new anisohedral pentagon

With Discovery, 3 Scientists Chip Away At An Unsolvable Math Problem

AUGUST 14, 2015 2:13 PM ET

4 days ago

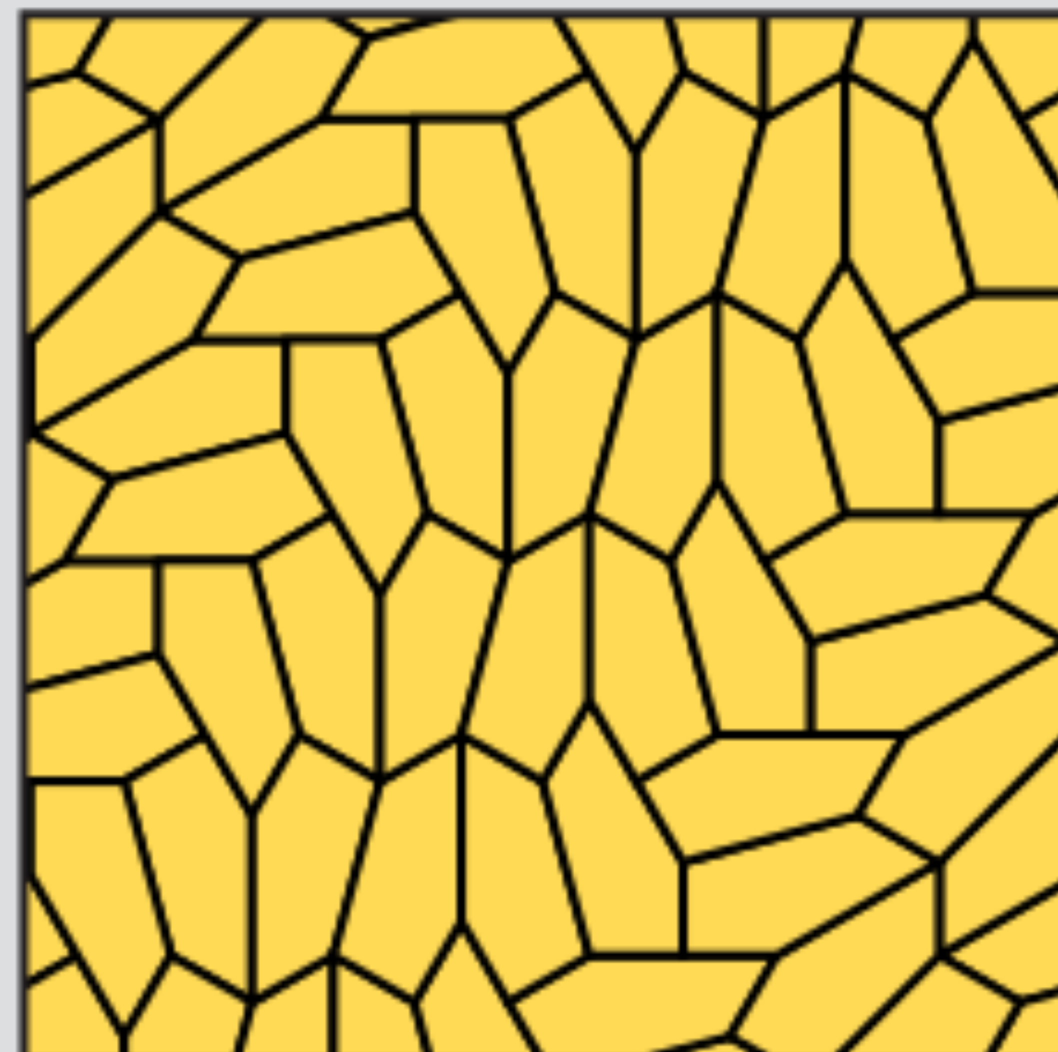
EYDER PERALTA



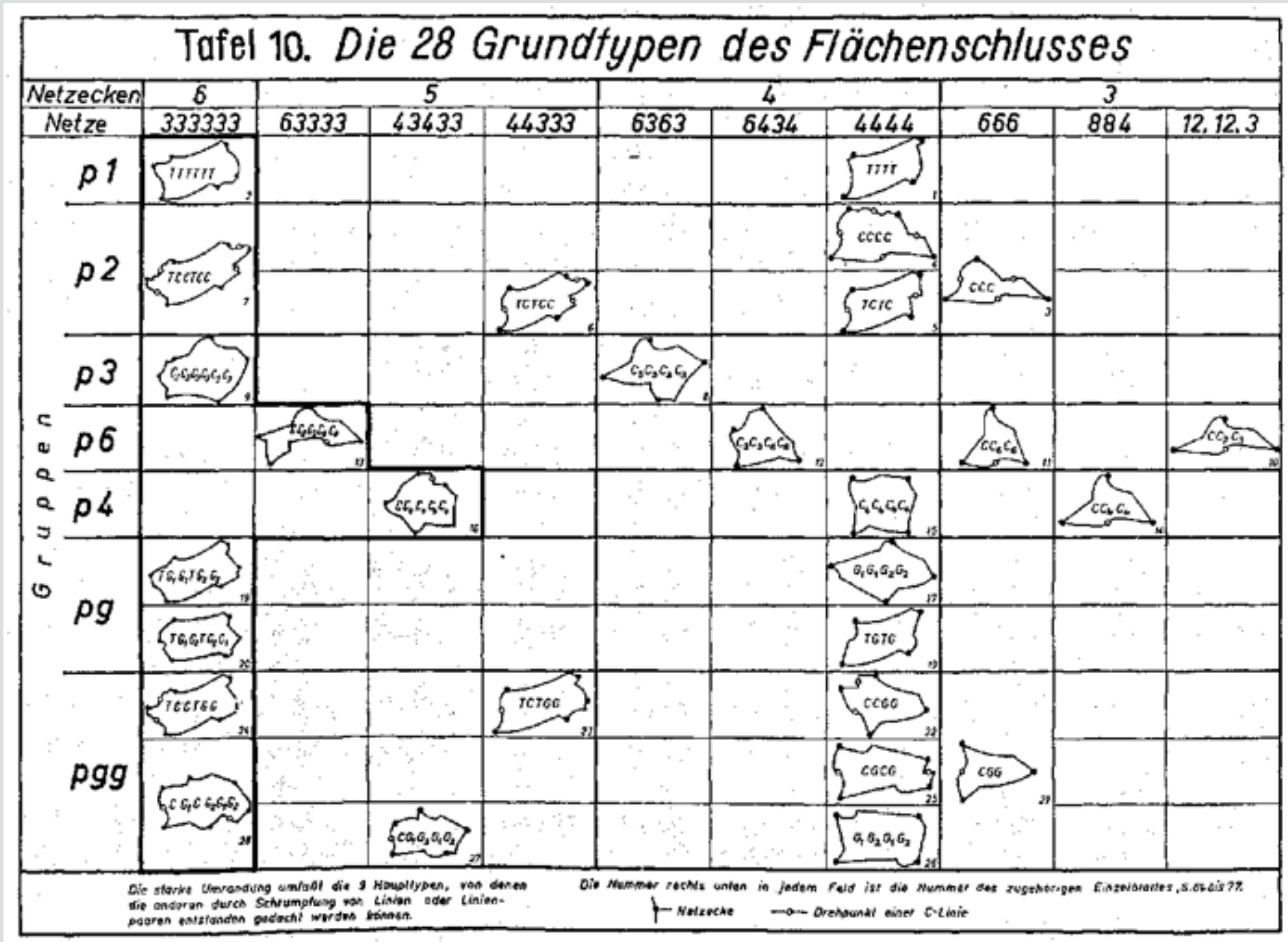
There are now 15 known convex pentagons, or nonregular pentagons with the angles pointing outward, that can "tile the plane."
EdPeggJr/Wikimedia Commons

Jennifer McLoud-Mann had almost come to believe that her last two years of work had been for naught.

"It had gotten to the point, where we hadn't found anything," she said. "And I was starting to believe I just don't know if we're going to find anything."

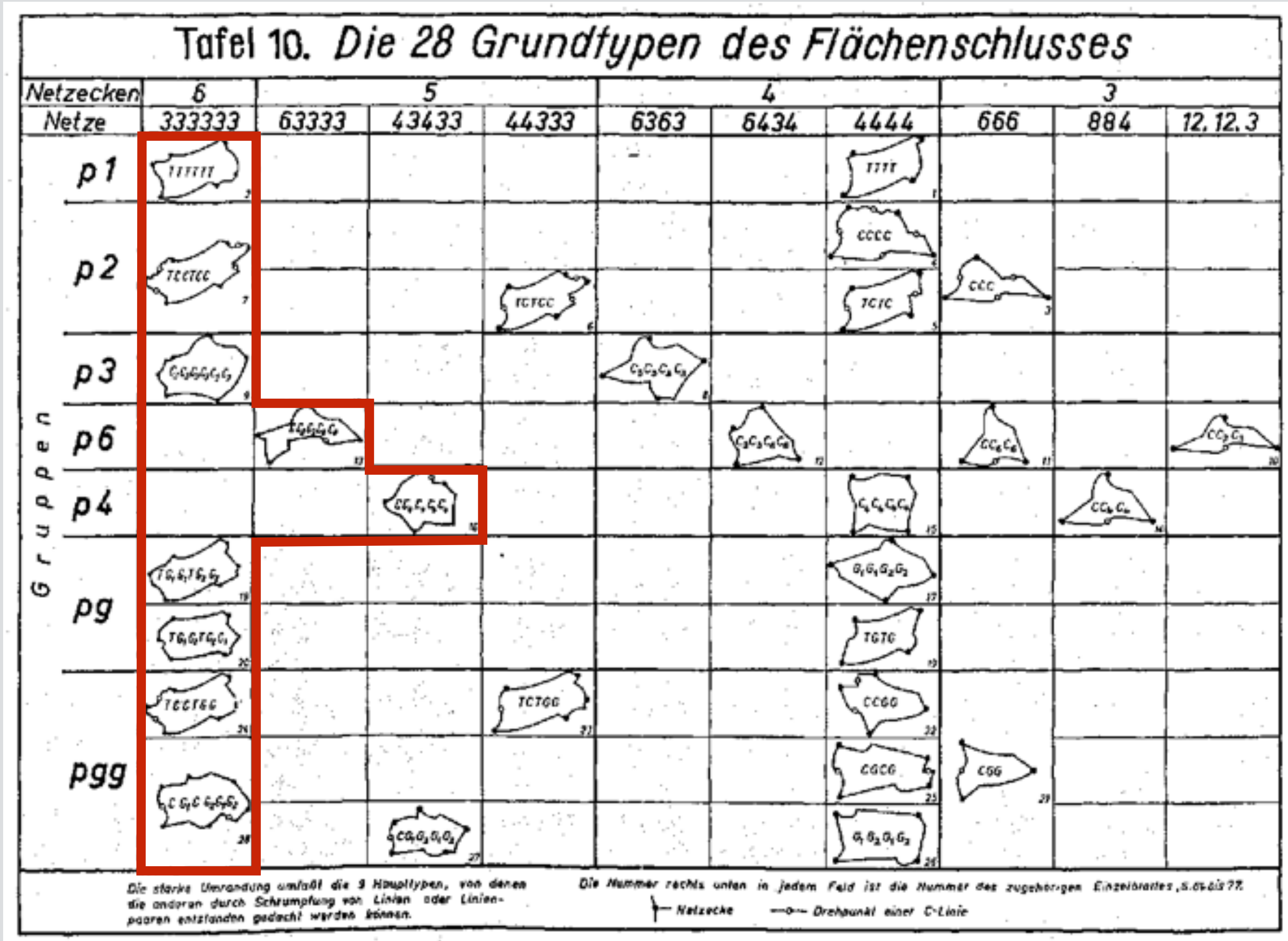


Isohedral boundary criteria



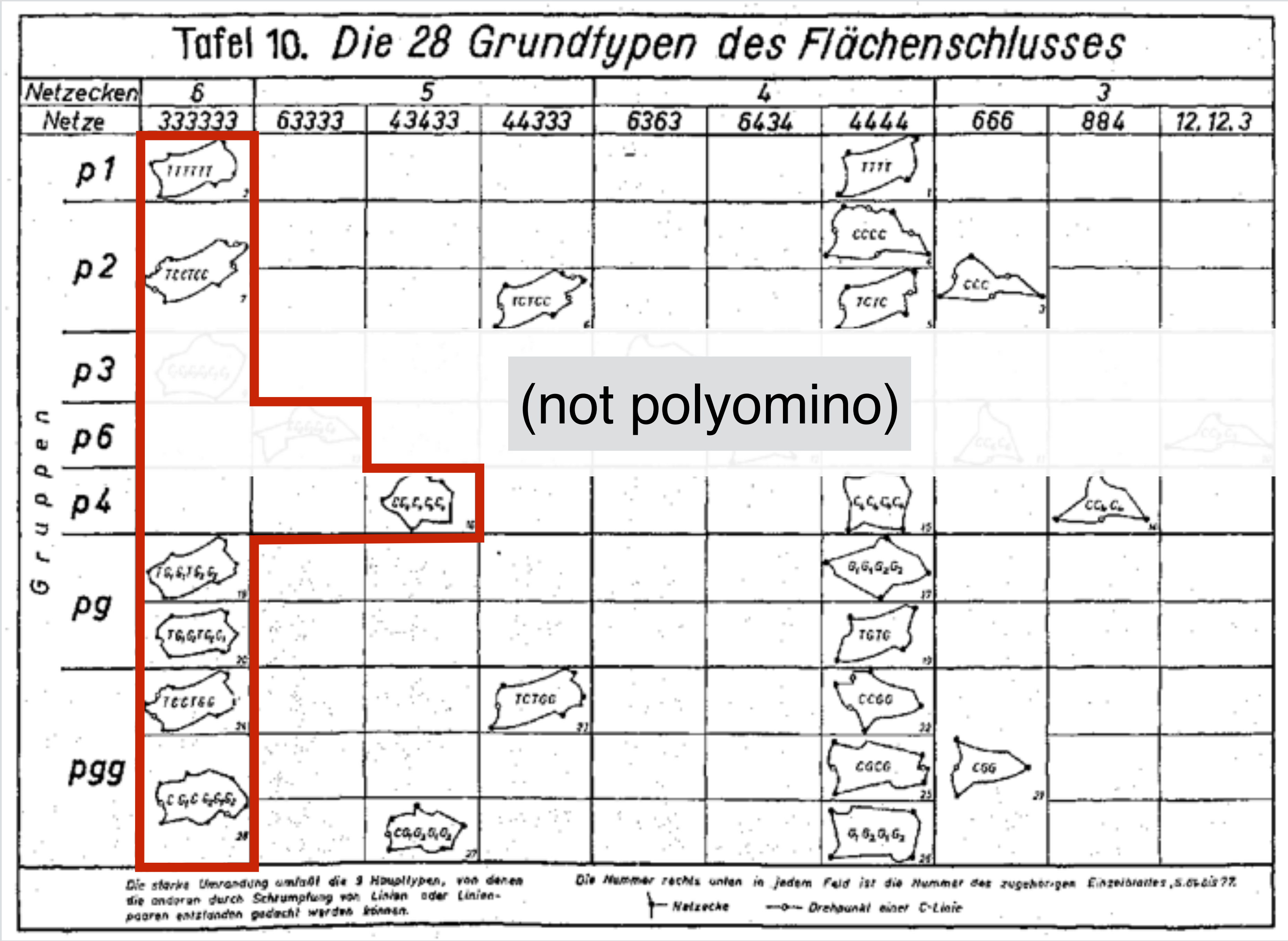
[Heesch, Kienzle 1963]

Isohedral boundary criteria



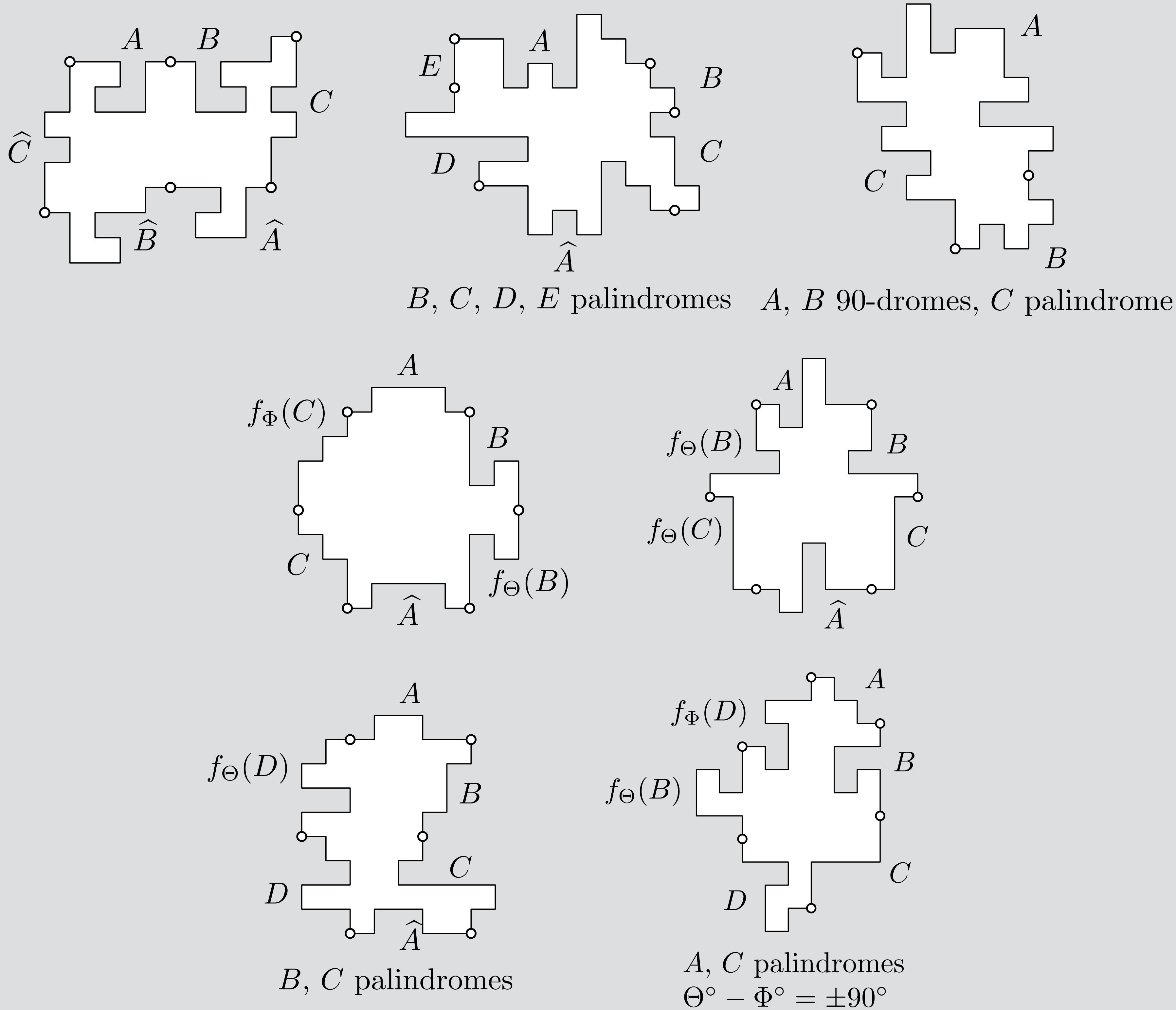
[Heesch, Kienzle 1963]

Isohedral boundary criteria



[Heesch, Kienzle 1963]

7 isohedral criteria



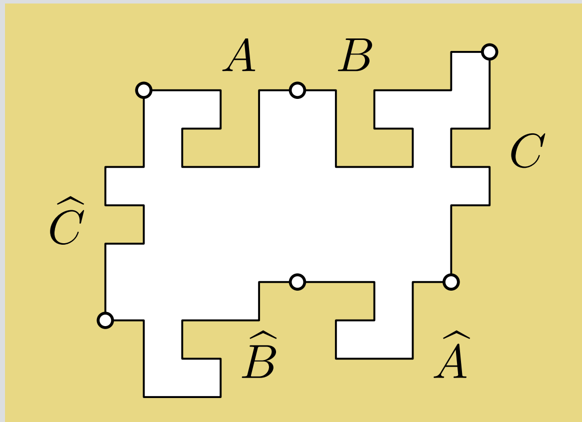
B, C, D, E palindromes

A, B 90-dromes, C palindrome

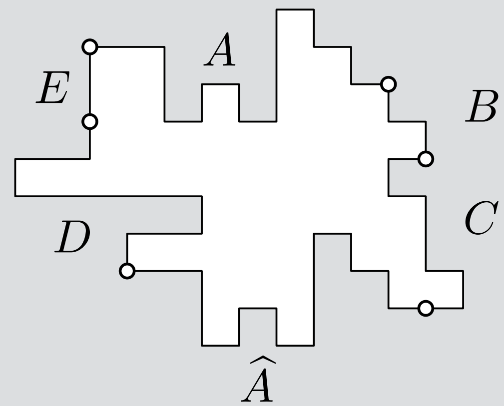
B, C palindromes

A, C palindromes
$\Theta^\circ - \Phi^\circ = \pm 90^\circ$

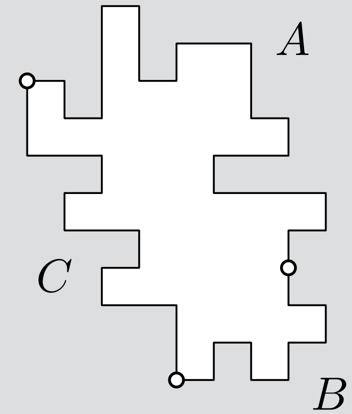
7 isohedral criteria



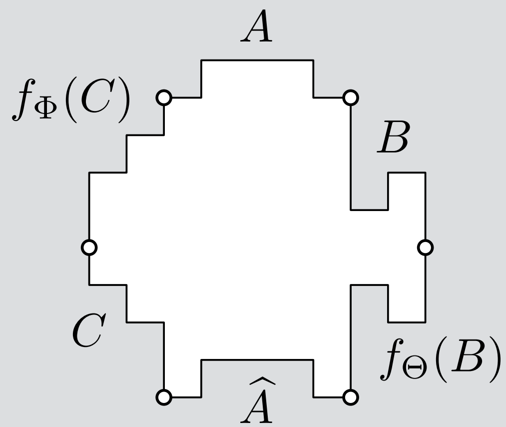
Translation criterion

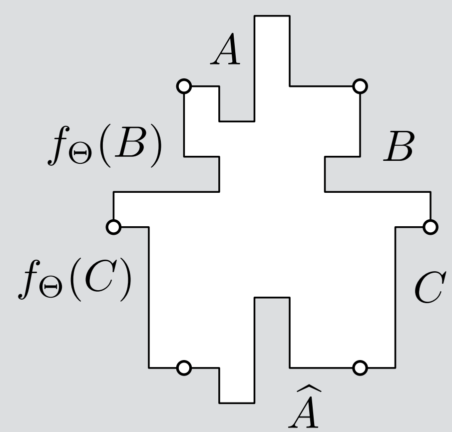


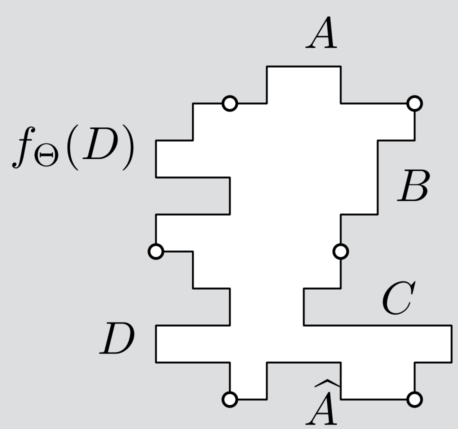
B, C, D, E palindromes



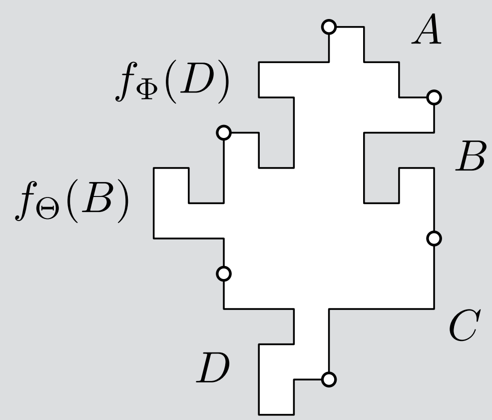
A, B 90-dromes, C palindrome





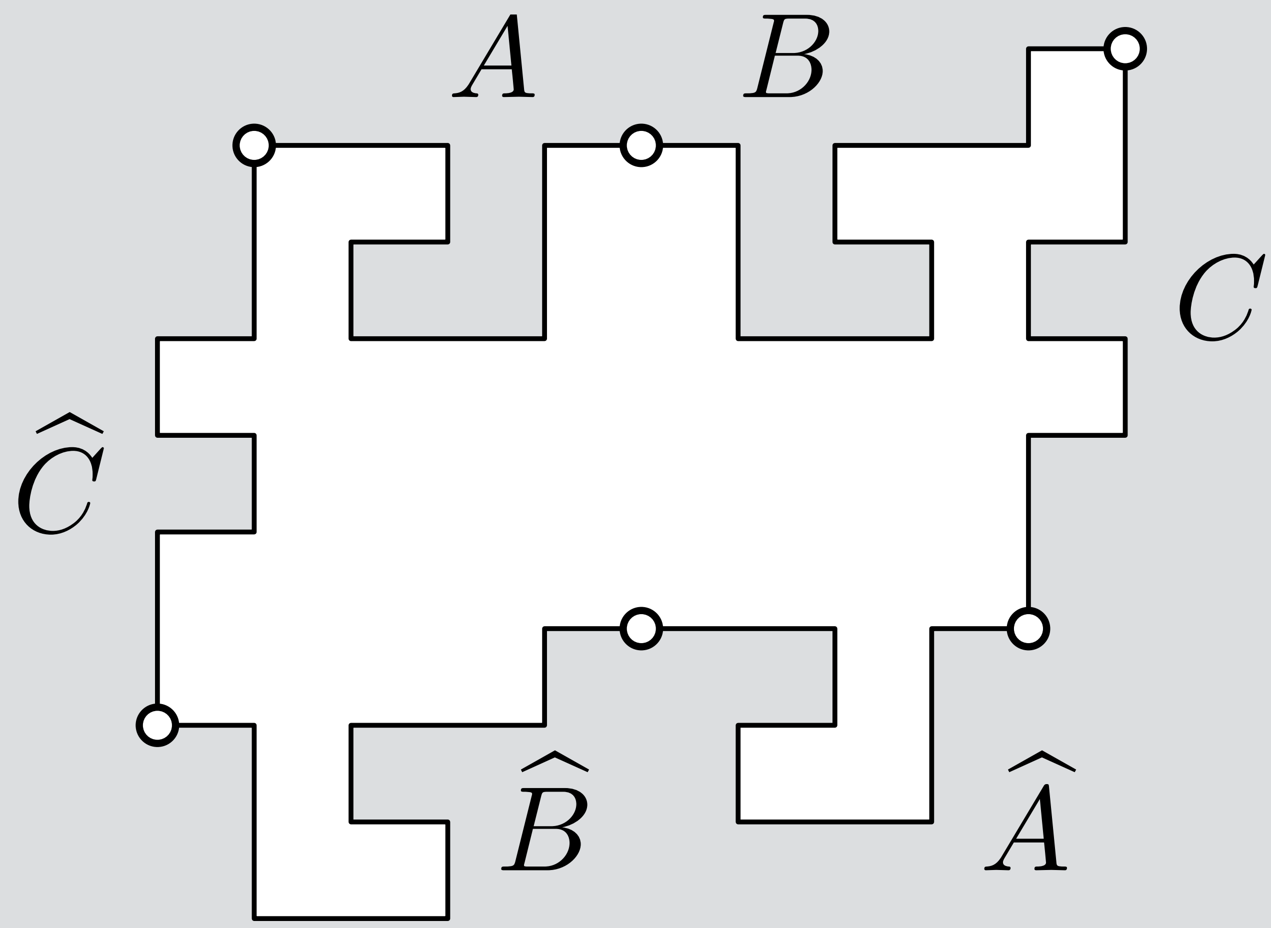


B, C palindromes



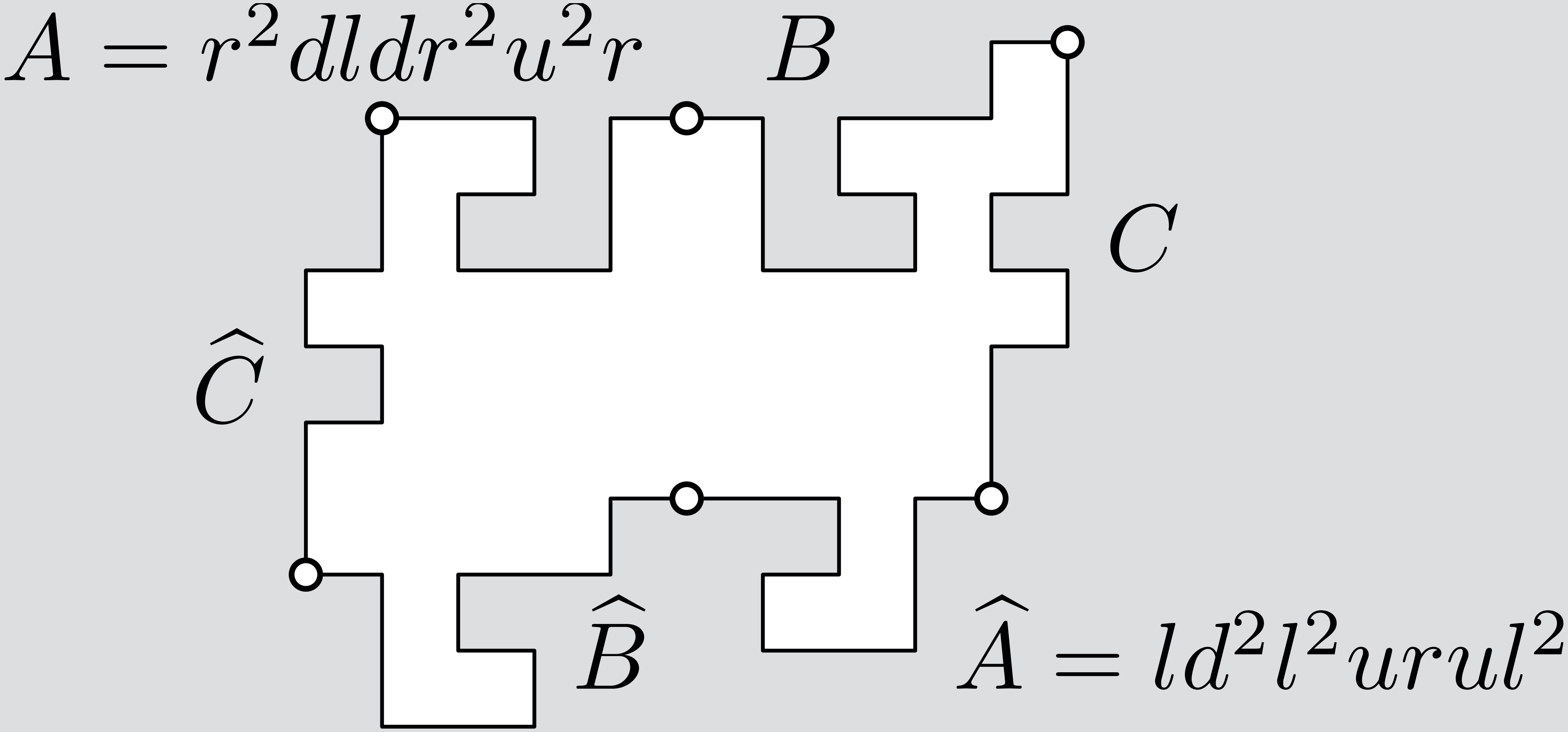
A, C palindromes
$\Theta^\circ - \Phi^\circ = \pm 90^\circ$

Translation criterion



$$X = x_1 x_2 \ldots x_n \qquad \overline{u} = d \quad \overline{r} = l$$
$$\widehat{X} = \overline{x}_n \overline{x}_{n-1} \ldots \overline{x}_1 \quad \text{with} \quad \overline{d} = u \quad \overline{l} = r$$

Translation criterion

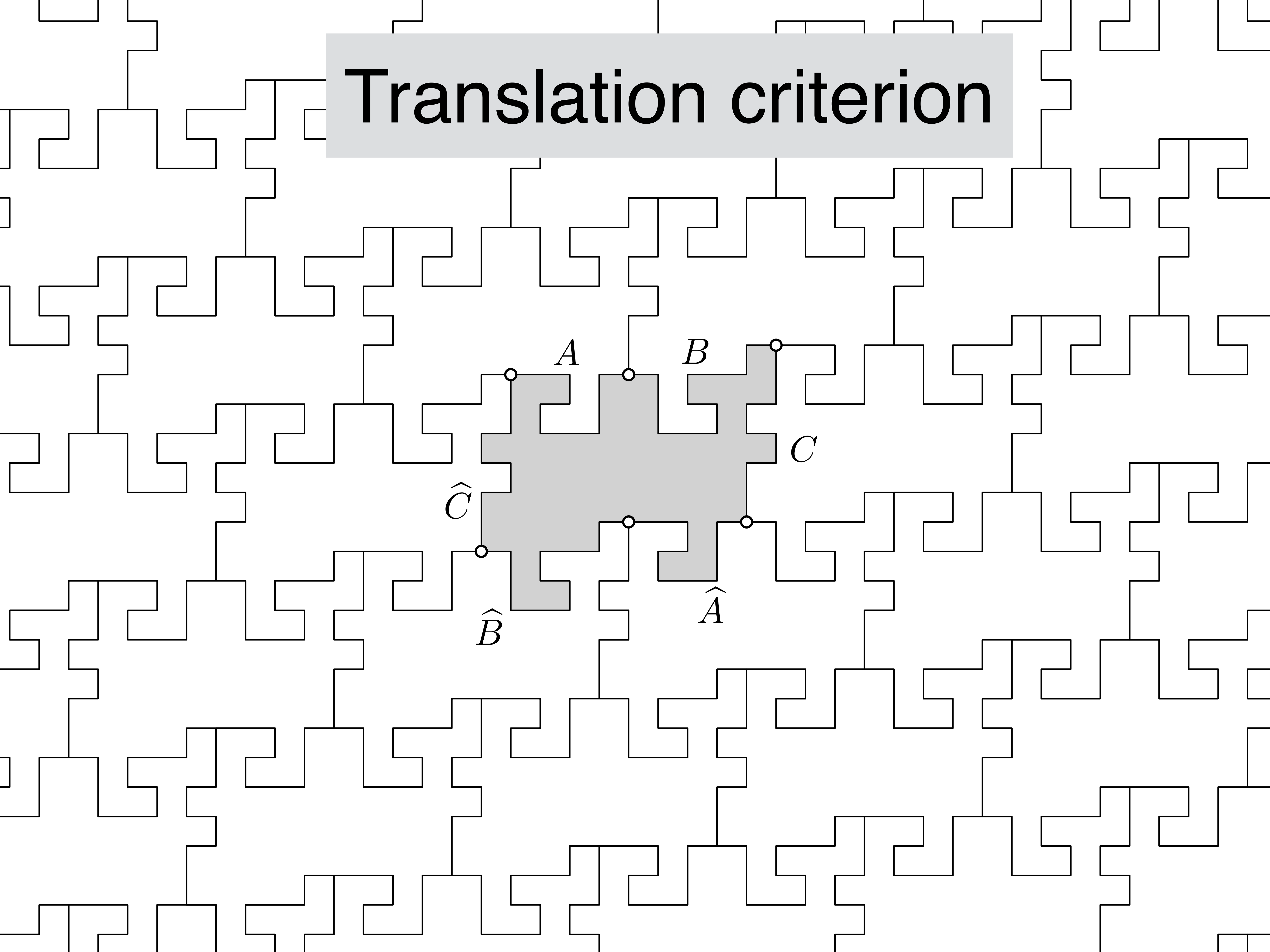


$$X = x_1x_2 \ldots x_n$$
$$\widehat{X} = \overline{x}_n\overline{x}_{n-1} \ldots \overline{x}_1$$

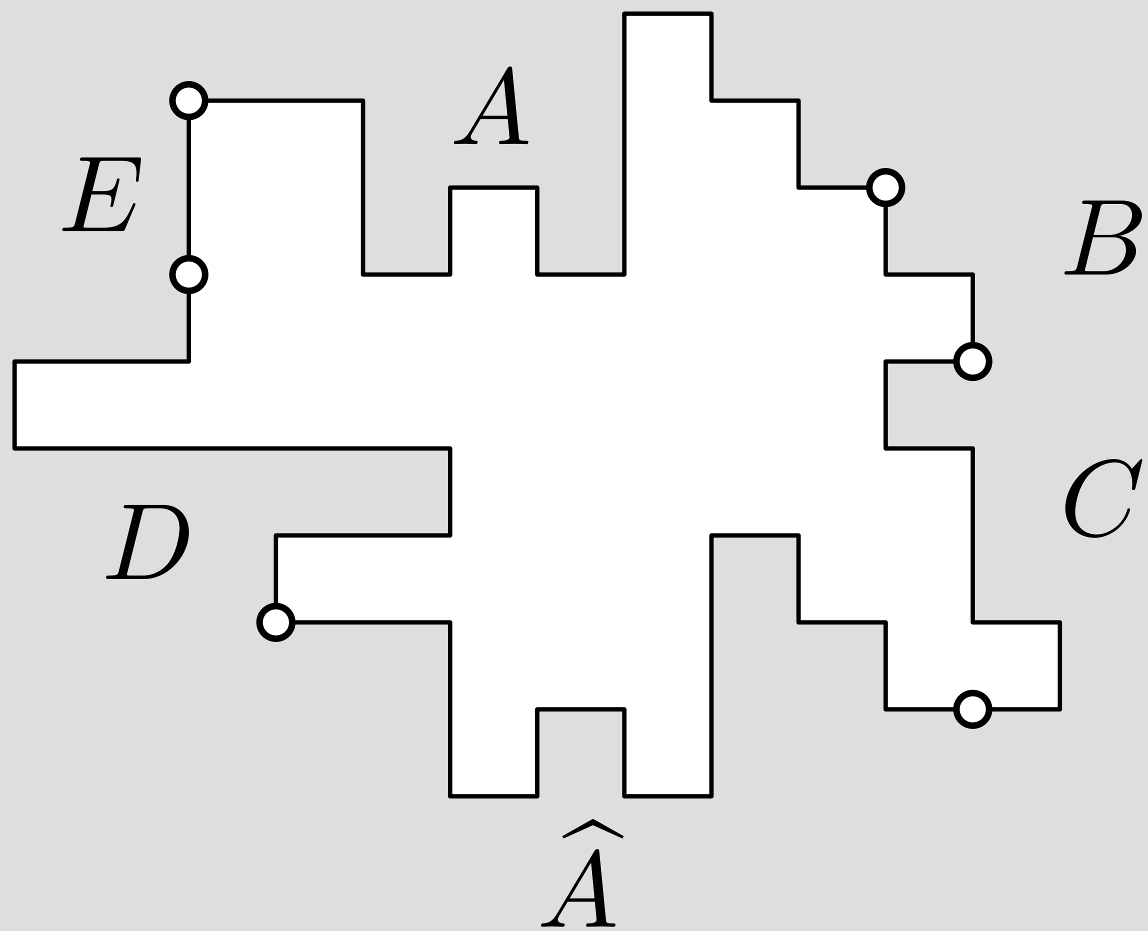
with

$$\overline{u} = d \quad \overline{r} = l$$
$$\overline{d} = u \quad \overline{l} = r$$

Translation criterion



Conway's criterion

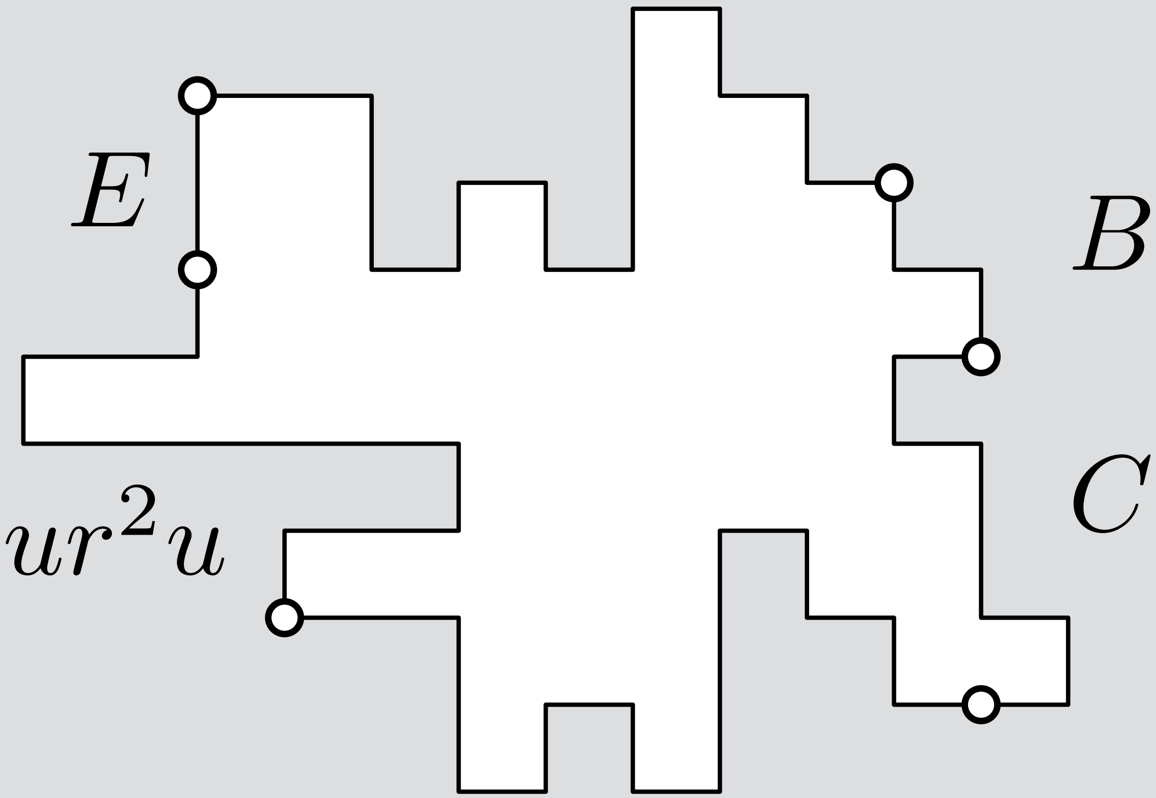


B, C, D, E palindromes

$$X = x_1 x_2 \ldots x_n \qquad \overline{u} = d \quad \overline{r} = l$$
with
$$\widehat{X} = \overline{x}_n \overline{x}_{n-1} \ldots \overline{x}_1 \qquad \overline{d} = u \quad \overline{l} = r$$

Conway's criterion

$$A = r^2 d^2 rurdru^3 r(dr)^2$$



$$D = ur^2 ul^5 ur^2 u$$

$$C = ldrd^2 rdl$$

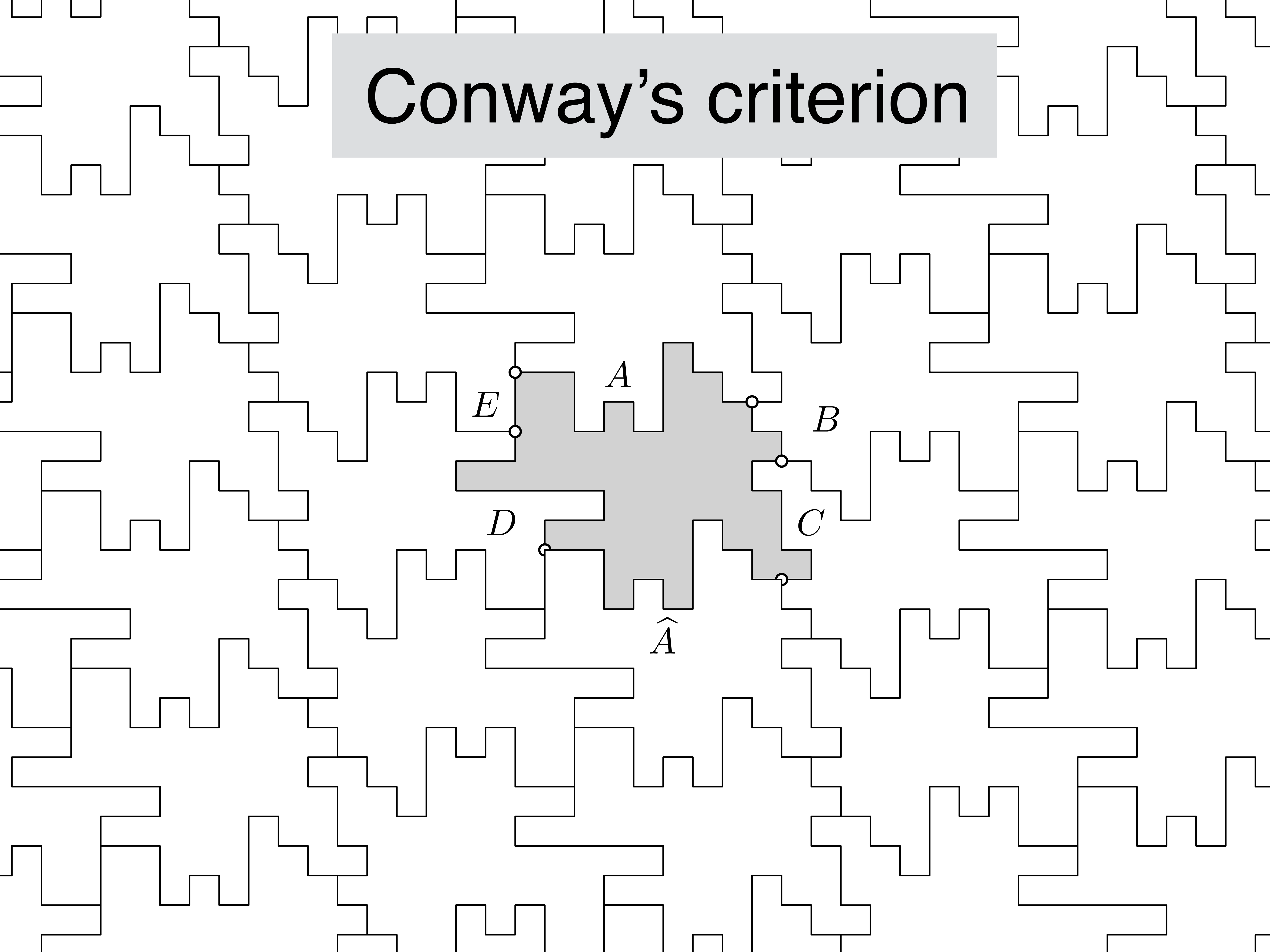
$$\widehat{A} = (lu)^2 ld^3 luldlu^2 l^2$$

B, C, D, E palindromes

$$X = x_1 x_2 \ldots x_n \qquad \overline{u} = d \quad \overline{r} = l$$

$$\widehat{X} = \overline{x}_n \overline{x}_{n-1} \ldots \overline{x}_1 \quad \text{with} \quad \overline{d} = u \quad \overline{l} = r$$

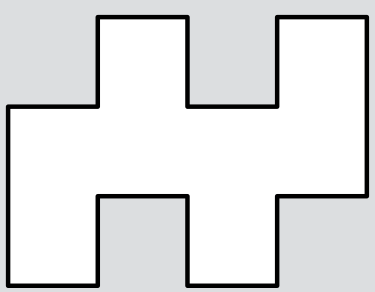
Conway's criterion



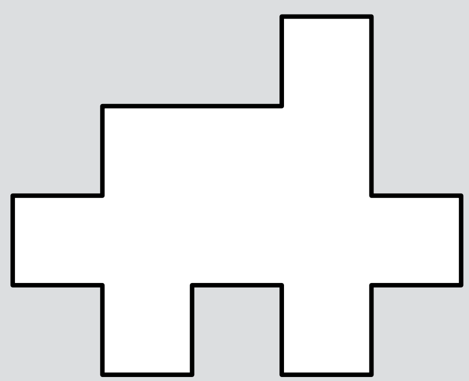
Problem

Decide whether a polyomino is isohedral.

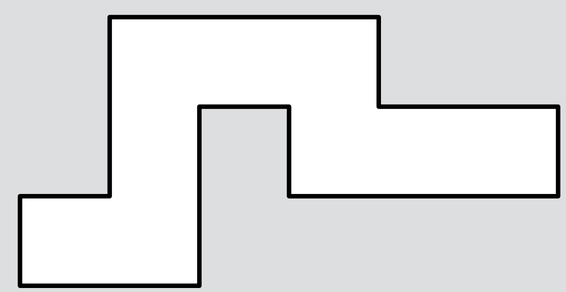
(passes any of 7 criteria)



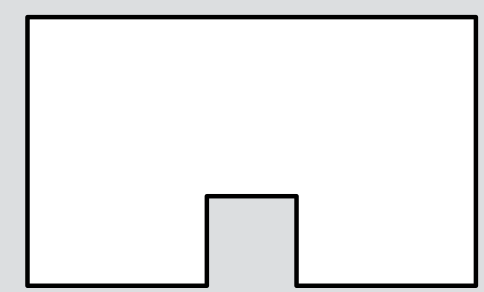
Yes.



Yes.



No.



No.

Problem: decide whether polyomino P with n sides is isohedral.

Problem: decide whether polyomino P with n sides is isohedral.

General case (all 7 criteria):

- [Keating, Vince 1999]: $O(n^{18})$
- Naive checking of criteria: $O(n^6)$
- [Langerman, W.]: $O(n*\log^2(n))$

Problem: decide whether polyomino P with n sides is isohedral.

General case (all 7 criteria):

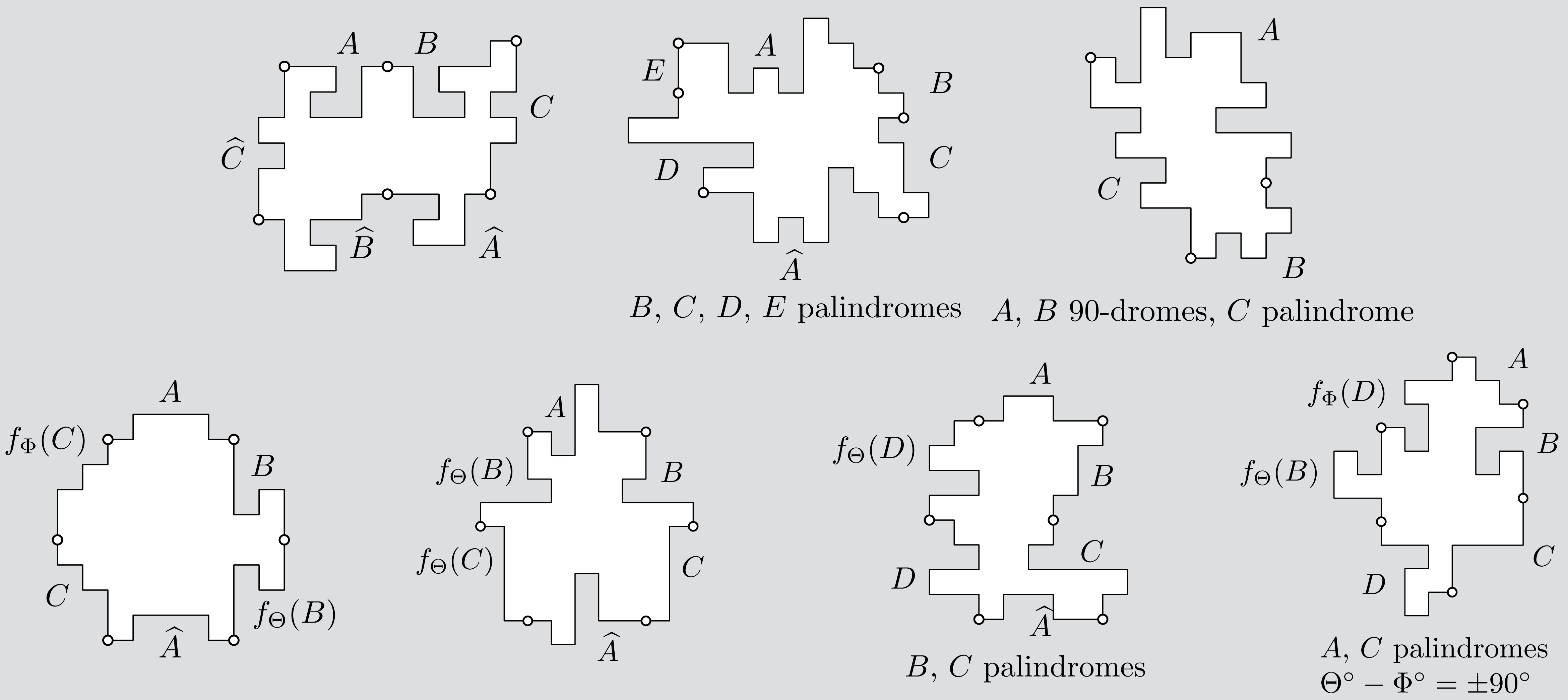
- [Keating, Vince 1999]: $O(n^{18})$
- Naive checking of criteria: $O(n^6)$
- [Langerman, W.]: $O(n*\log^2(n))$

Translation criterion only:

- [Gambini, Vuillon 2007]: $O(n^2)$
- [Provençal 2008]: $O(n*\log^3(n))$
- [Brlek, Provençal, Fédou 2009]: $O(n)$ (special cases)
- [W. 2015]: $O(n)$

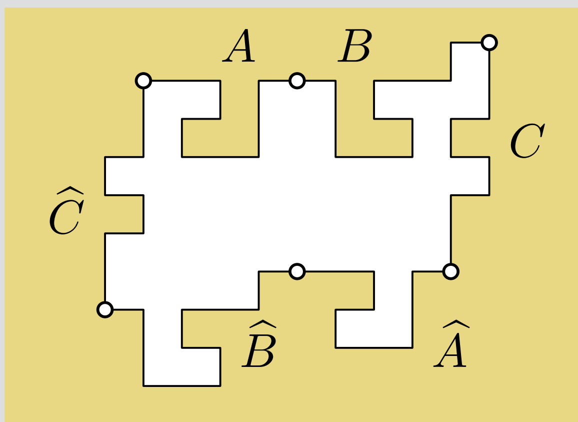
Algorithm

Test the input boundary for each criterion, using structural and algorithmic results on words.

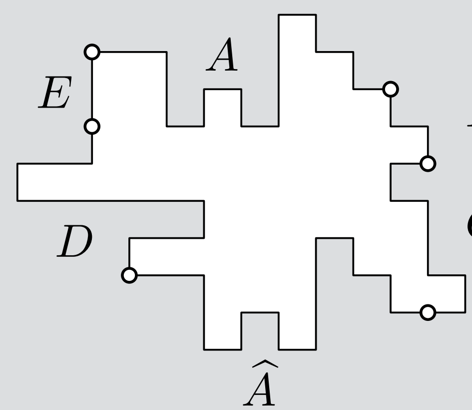


Algorithm

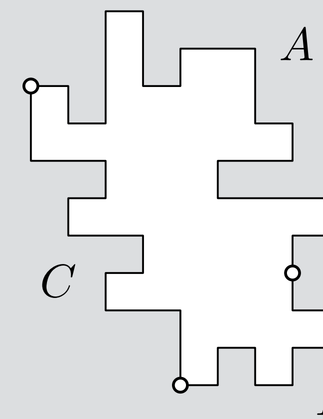
Test the input boundary for each criterion,
using structural and algorithmic results on words.



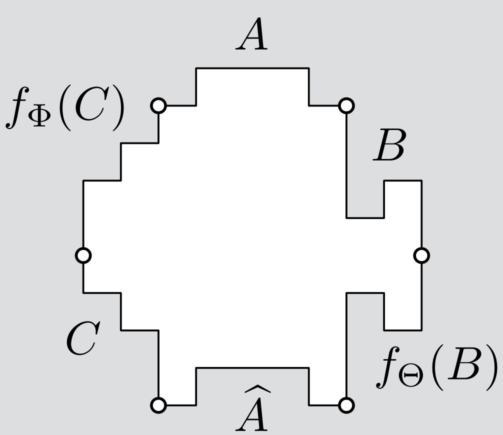
Translation criterion

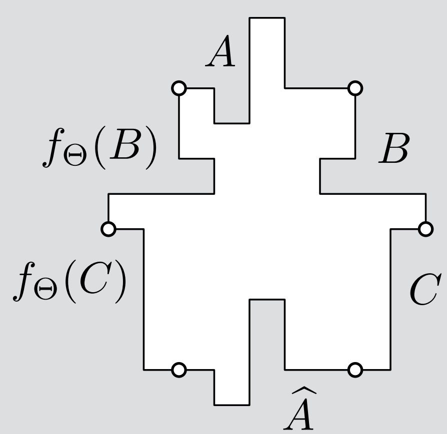


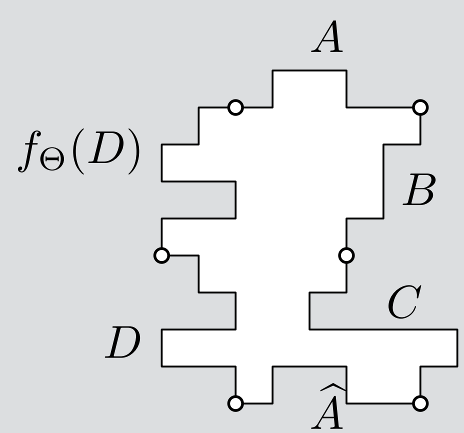
B, C, D, E palindromes



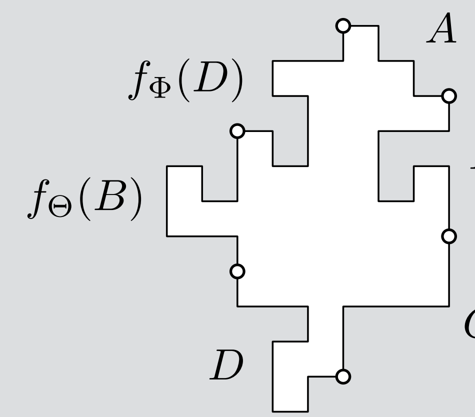
A, B 90-dromes, C palindrome





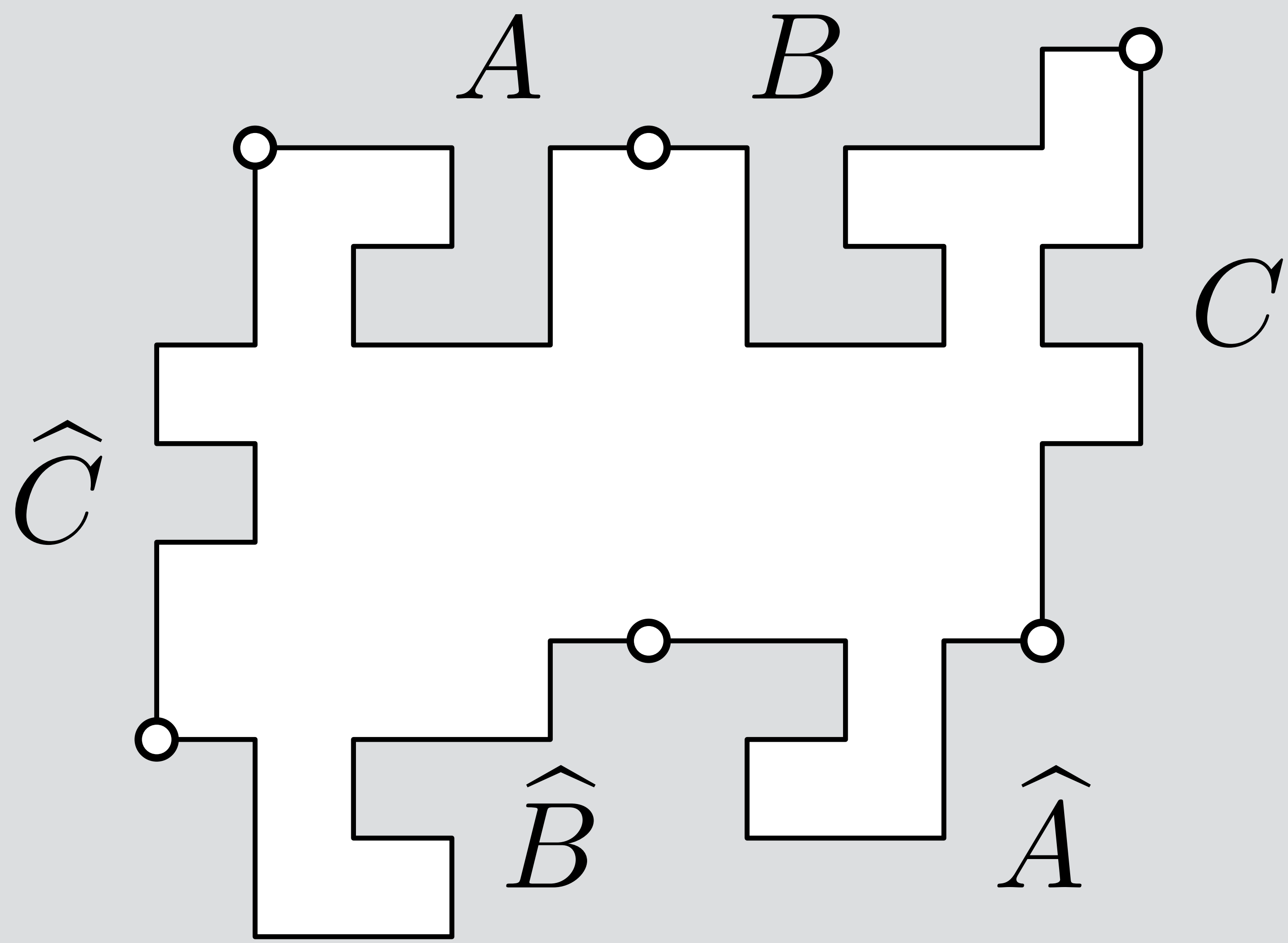


B, C palindromes



A, C palindromes
$\Theta^\circ - \Phi^\circ = \pm 90^\circ$

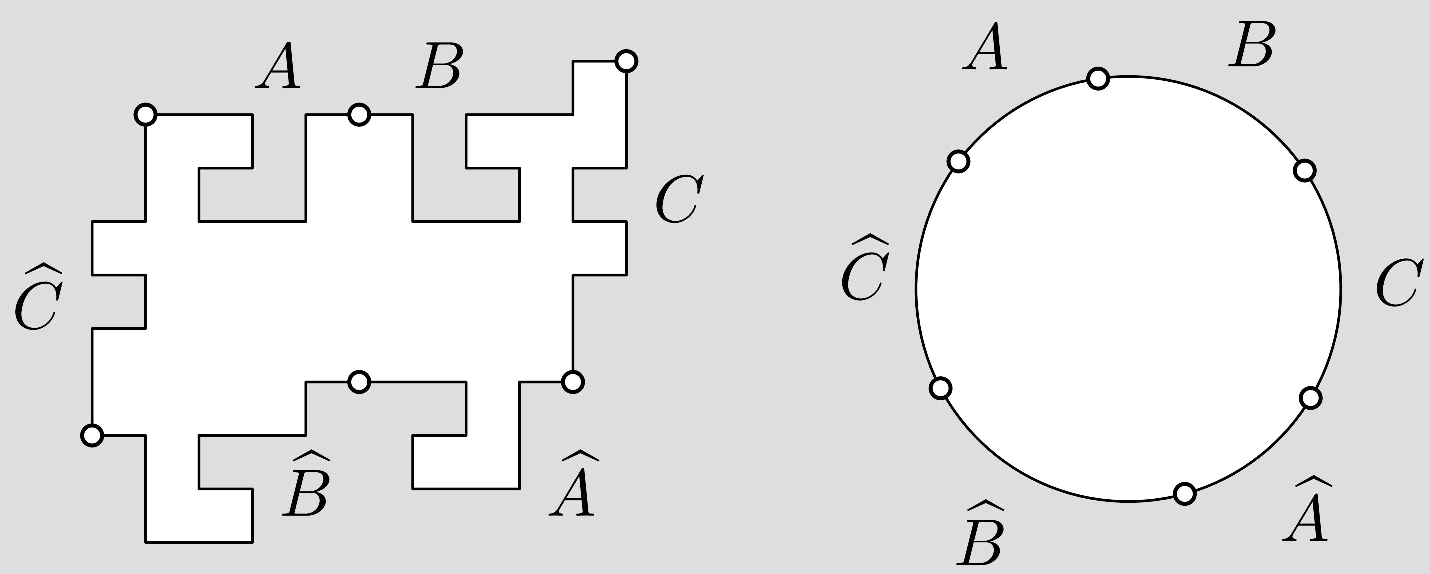
Testing for translation factorization



Decide if an input boundary word W has a translation factorization $W = ABC\widehat{A}\widehat{B}\widehat{C}$.

Testing for translation factorization

Step 1: compute all admissible factors.



For every factor A: $W = AU\widehat{A}V$ with $|U| = |V|$

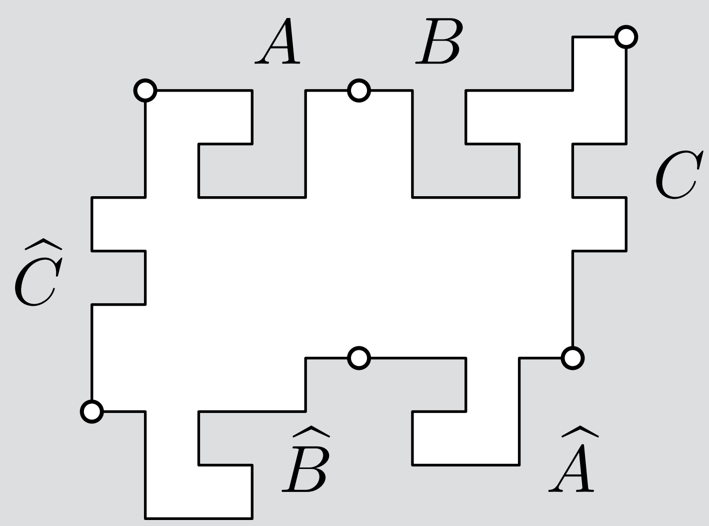
[Brlek et al. 2009]: $U[1] \neq \overline{U[-1]},\ V[1] \neq \overline{V[-1]}$ +

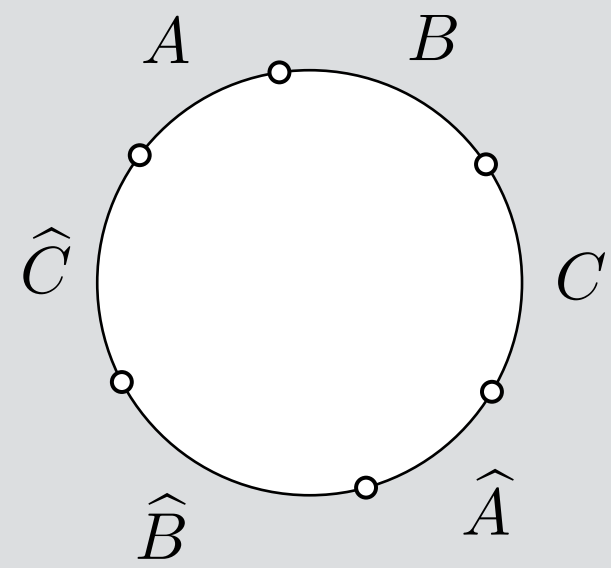
Call these factors <u>admissible</u>.

Can compute all 2n admissible factors in O(n) time.

Testing for translation factorization

Can compute all 2n admissible factors in O(n) time.





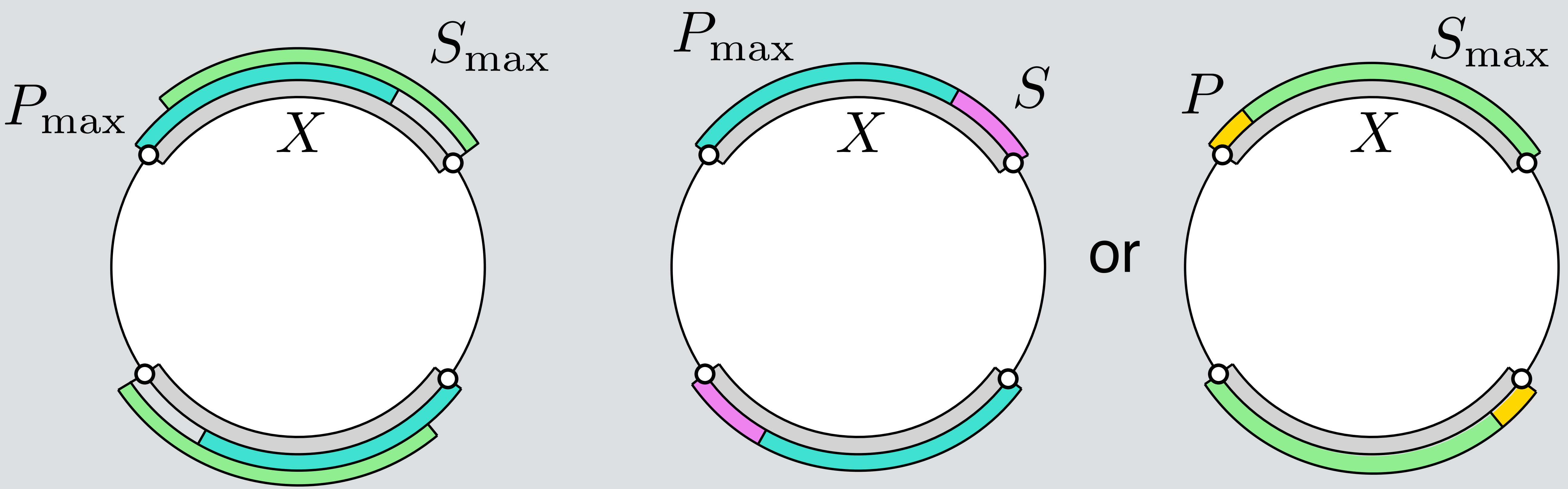
Factorization exists iff admissible factors A,B,C that are consecutive with |ABC| = n/2.

“Solution”: for each choice of A, look for B, C with |ABC| = n/2 (in O(1) time).

Lemma: X has factorization into two admissible factors if and only if X = P_{max}S or X = PS_{max} with:

- P_{max} the longest prefix admissible factor of X, or
- S_{max} the longest suffix admissible factor of X.

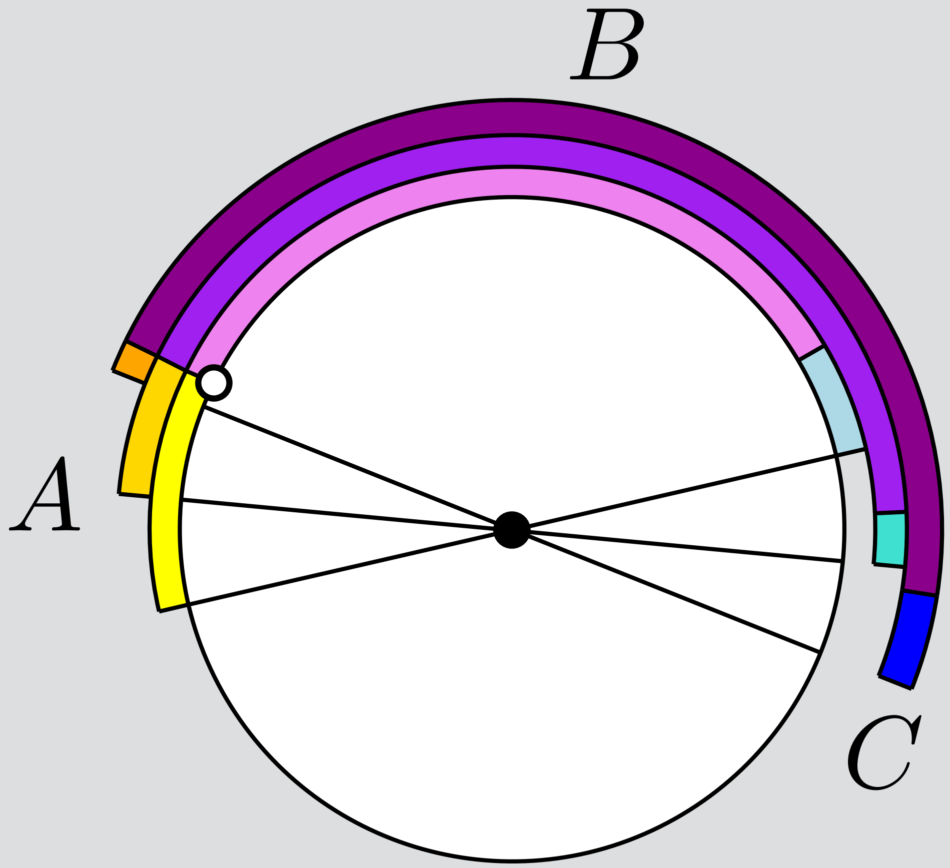
and P, S admissible factors.

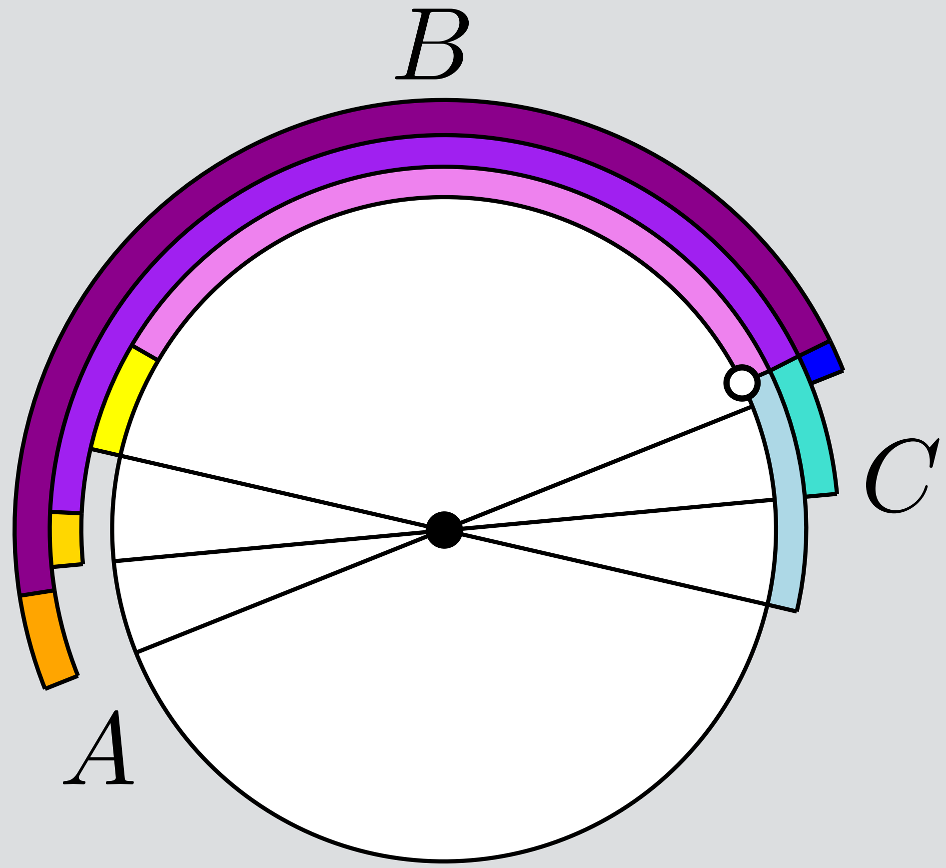


Proof follows that of similar result by [Galil, Seiferas 1978]

Finding consecutive A,B,C with |ABC| = n/2.

- For each A, search for longest B such that |AB| ≤ n/2, check whether factor C with |ABC| = n/2 is admissible.
- For each C, search for longest B such that |BC| ≤ n/2, check whether factor A with |ABC| = n/2 is admissible.



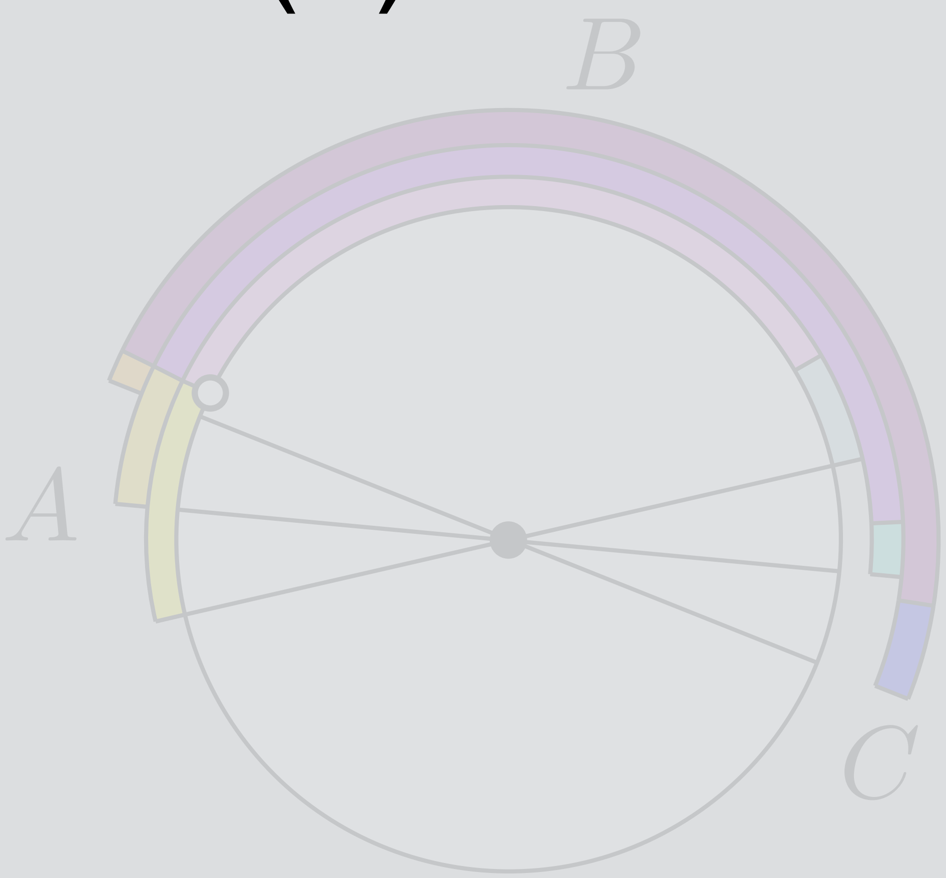


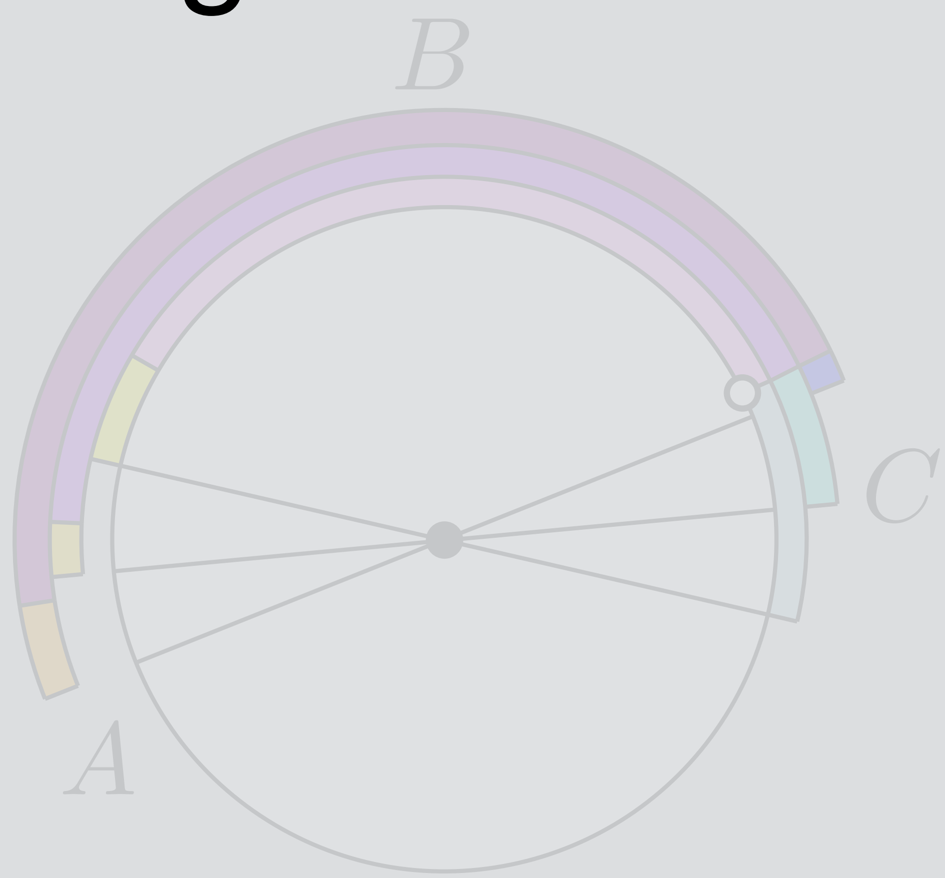
Finding consecutive A,B,C with |ABC| = n/2.

- For each A, search for longest B such that |AB| ≤ n/2, check whether factor C with |ABC| = n/2 is admissible.

- For each C, search for longest B such that |BC| ≤ n/2, check whether factor A with |ABC| = n/2 is admissible.

O(n) time using two-finger scans.





Testing for translation factorization

1. Compute all 2n admissible factors.

2. Sort admissible factors starting at each letter.
   Repeat for ending at each letter.

3. Two-finger scans to search for A,B,C
   that are consecutive with $|ABC| = n/2$.

Testing for translation factorization

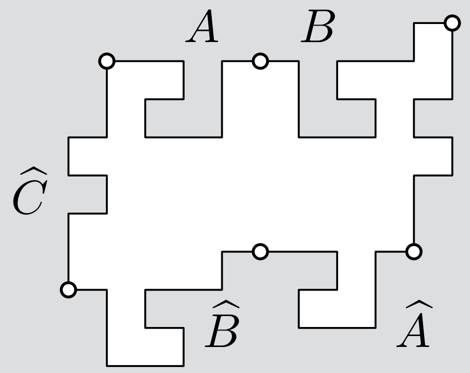
1. Compute all 2n admissible factors.

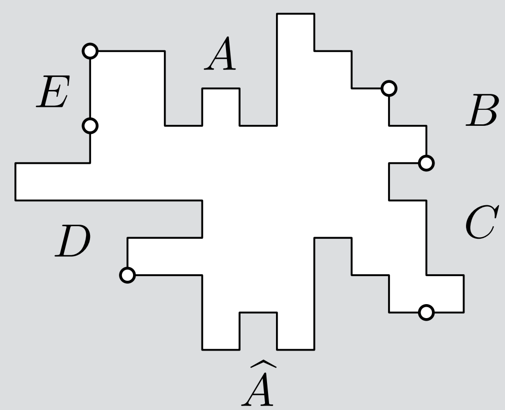
2. Sort admissible factors starting at each letter.
   Repeat for ending at each letter.

3. Two-finger scans to search for A,B,C
   that are consecutive with $|ABC| = n/2$.

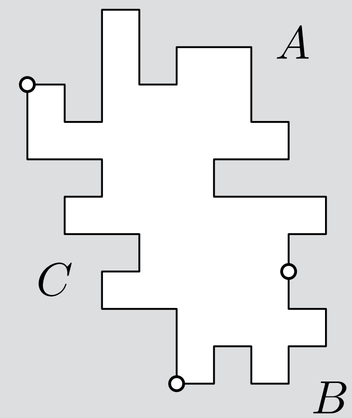
O(n) time for each step, O(n) total time.

Algorithm running times

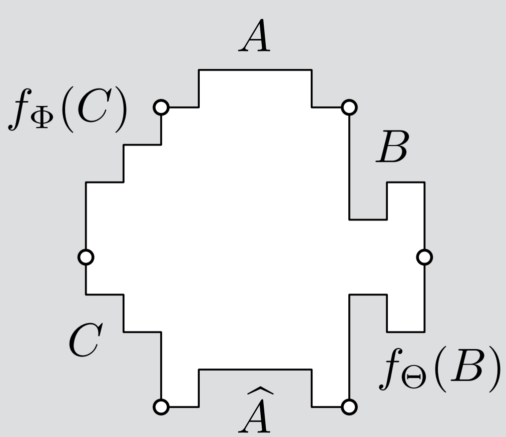


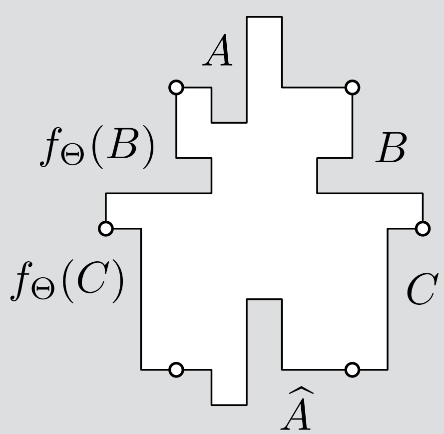


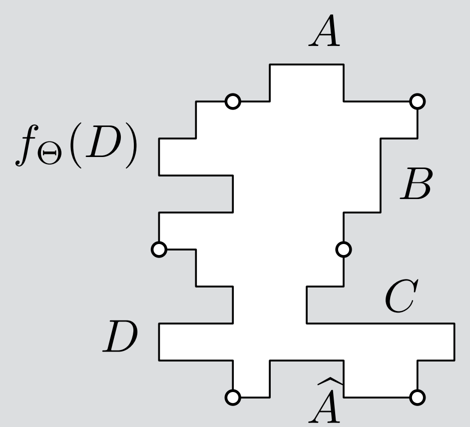
B, C, D, E palindromes



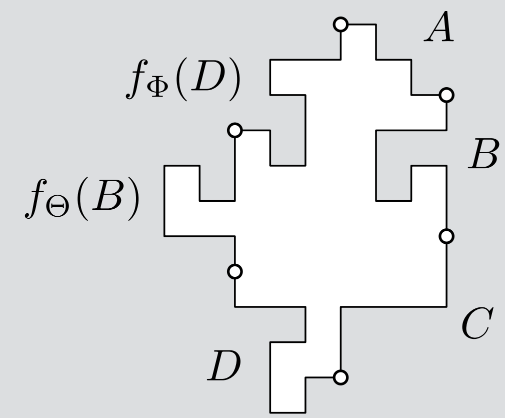
A, B 90-dromes, C palindrome





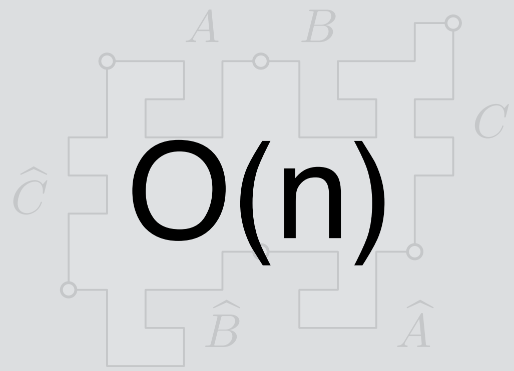


B, C palindromes

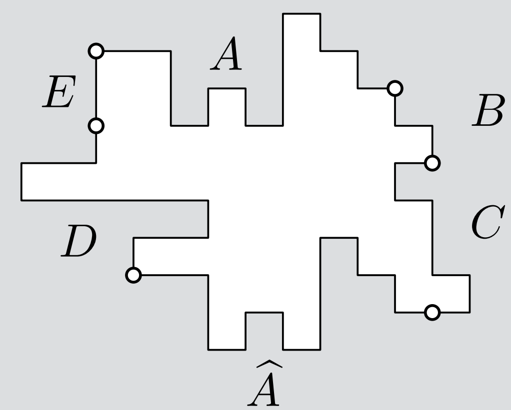


A, C palindromes
$\Theta^\circ - \Phi^\circ = \pm 90^\circ$

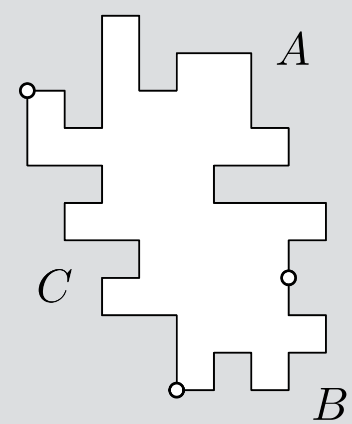
Algorithm running times



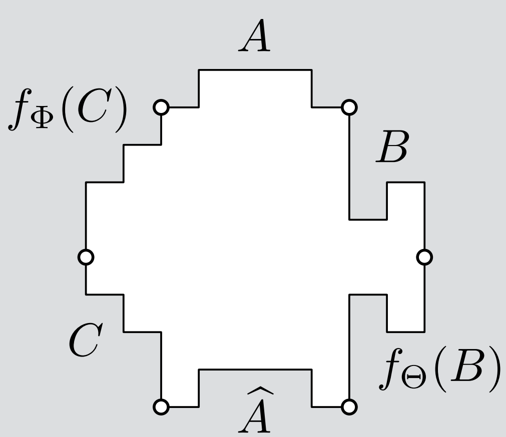
O(n)

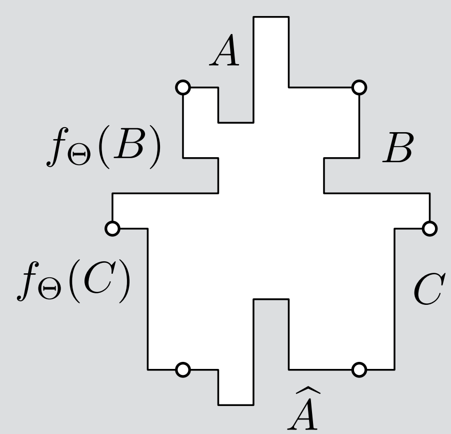


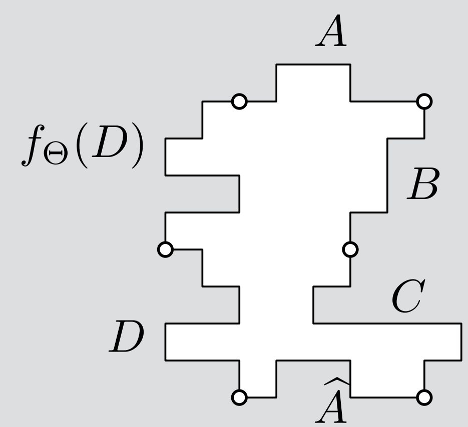
B, C, D, E palindromes



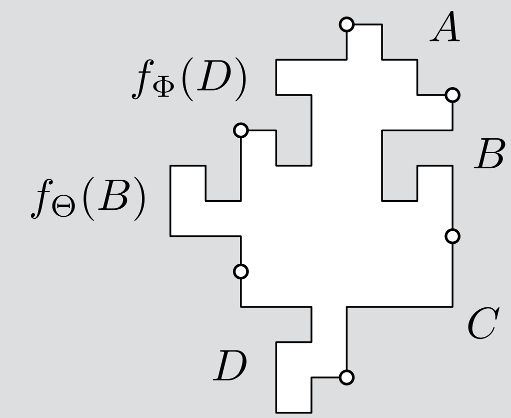
A, B 90-dromes, C palindrome





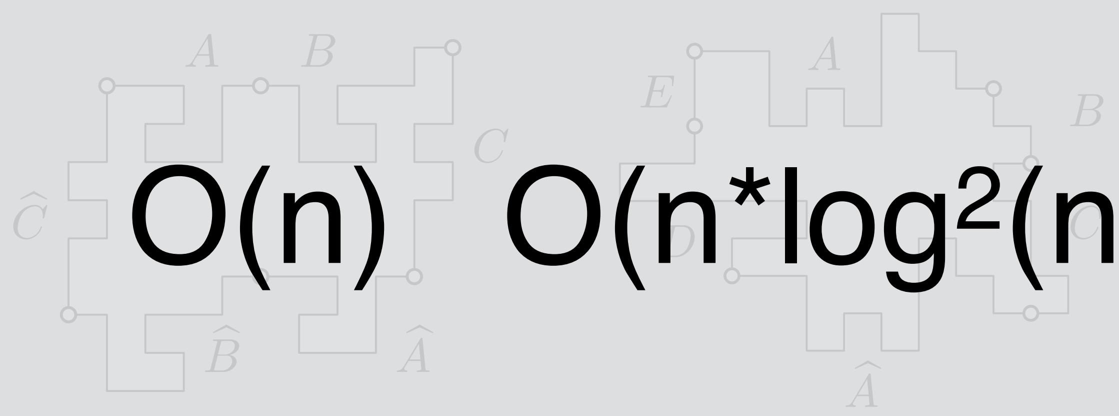


B, C palindromes



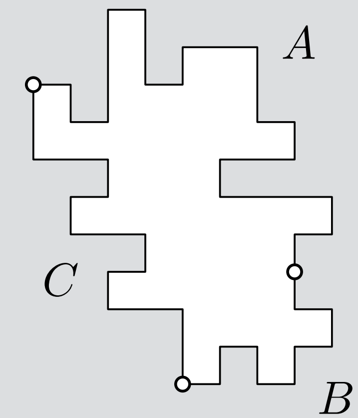
A, C palindromes
$\Theta^\circ - \Phi^\circ = \pm 90^\circ$

Algorithm running times

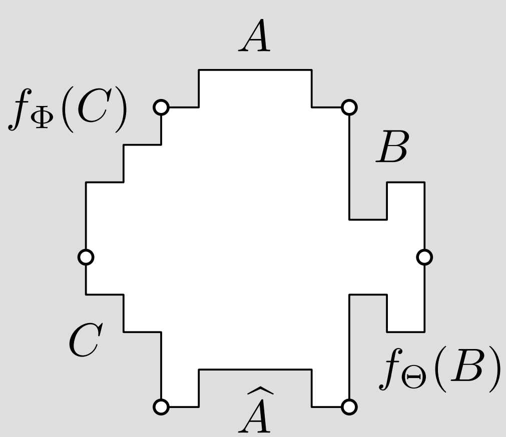


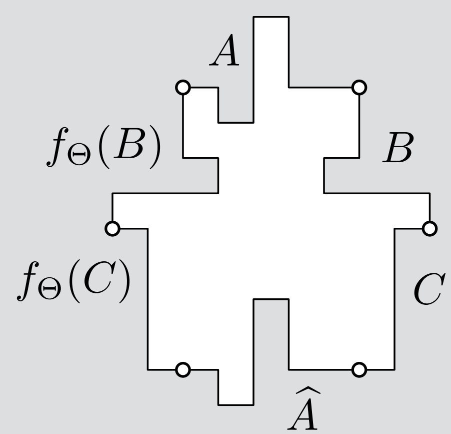
B, C, D, E palindromes

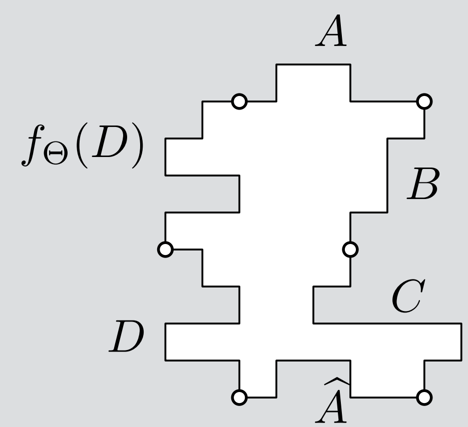
O(n)   O(n*log²(n))



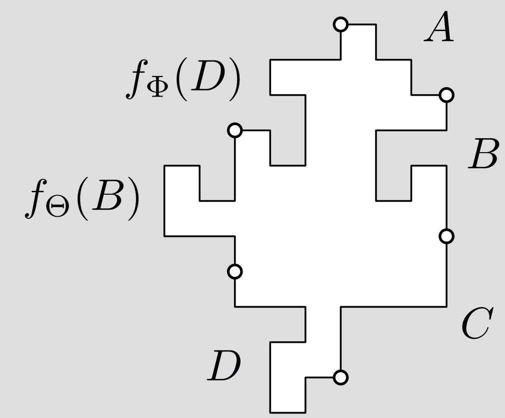
A, B 90-dromes, C palindrome





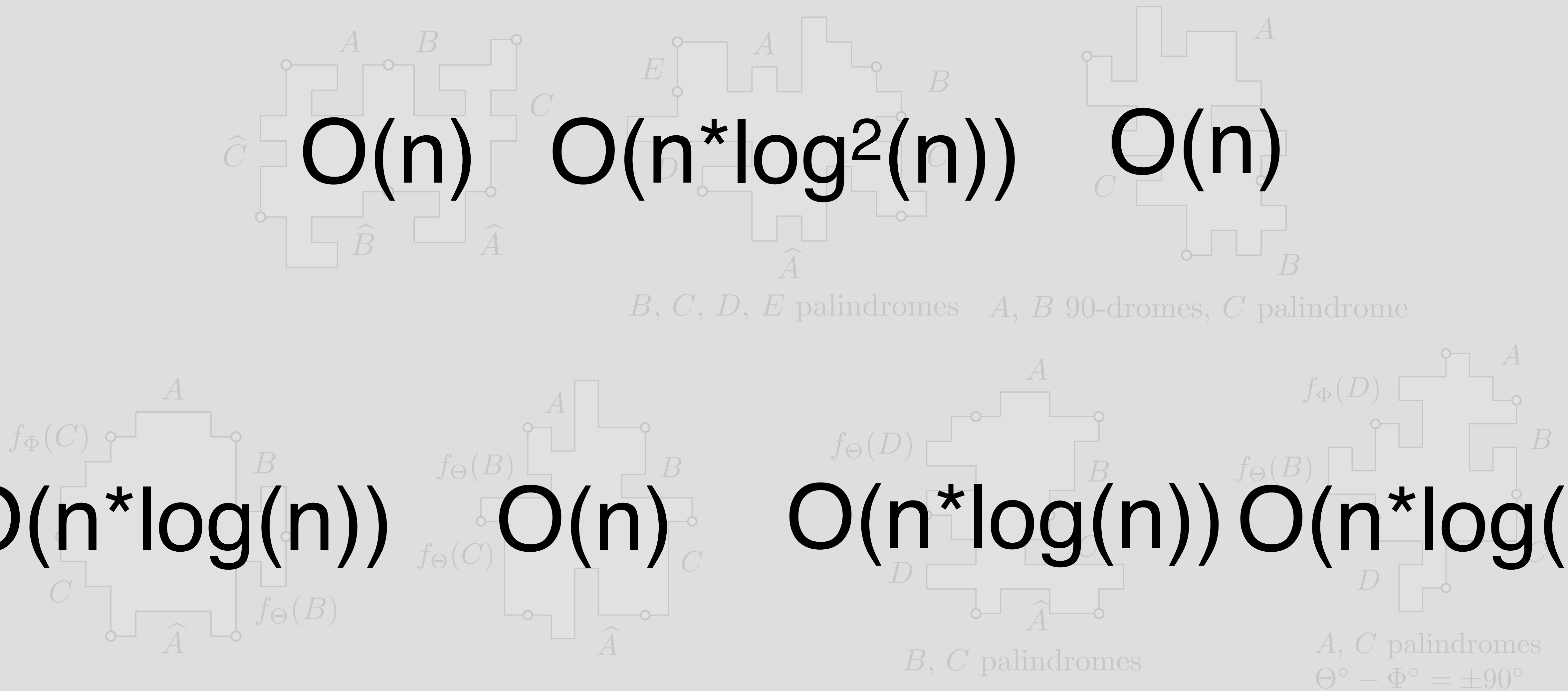


B, C palindromes



A, C palindromes
$\Theta^\circ - \Phi^\circ = \pm 90^\circ$

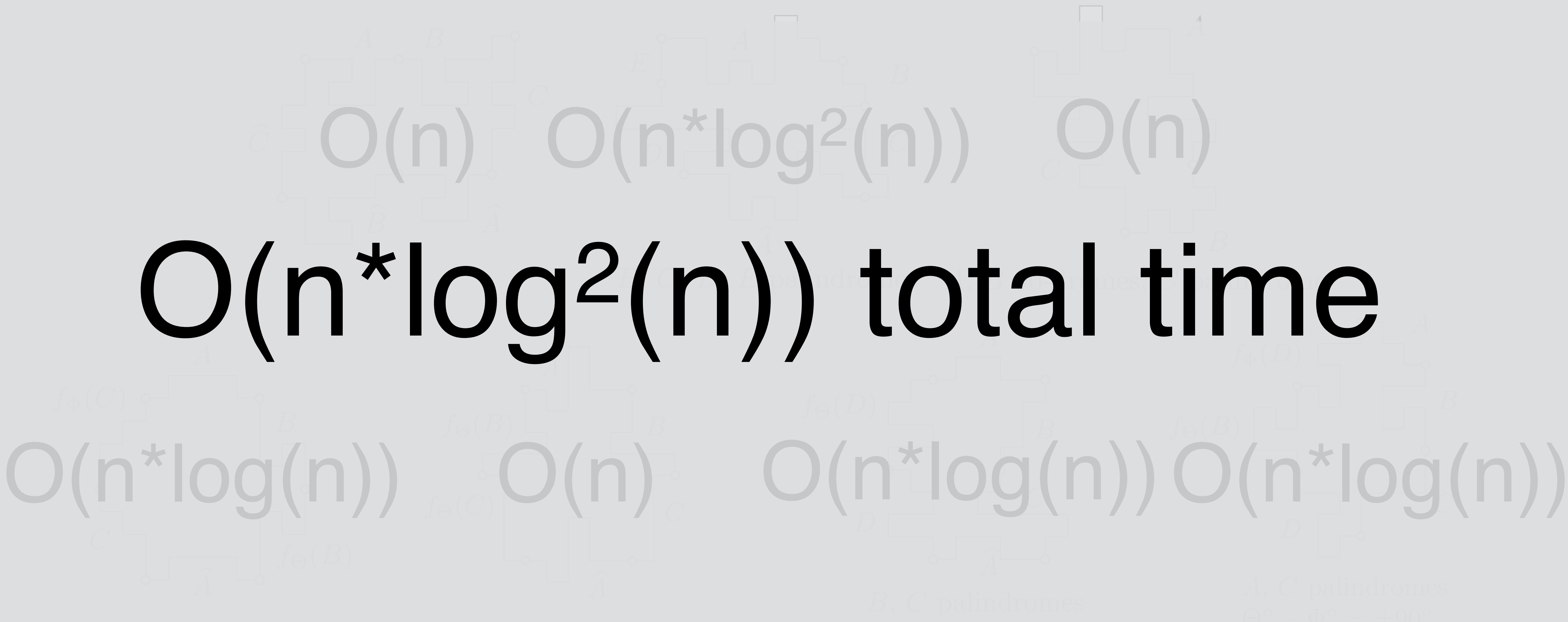
Algorithm running times



O(n) O(n*log²(n)) O(n)

O(n*log(n)) O(n) O(n*log(n)) O(n*log(n))

Algorithm running times



O($n*\log^2(n)$) total time

Open Problems

Known: $O(n*\log^2(n))$-time algorithm for deciding if a polyomino tiles the plane isohedrally.

Open Problems

Known: $O(n*\log^2(n))$-time algorithm for deciding if a polyomino tiles the plane isohedrally.

$O(n)$-time algorithm?

Open Problems

Known: O(n*log^2(n))-time algorithm for deciding if a polyomino tiles the plane isohedrally.

O(n)-time algorithm?

Enumeration of tilings in O(n*log^2(n) + k) time?

Open Problems

Known: $O(n*\log^2(n))$-time algorithm for deciding if a polyomino tiles the plane isohedrally.

O(n)-time algorithm?

Enumeration of tilings in O(n*log^2(n) + k) time?

Extend inputs to polygons?

Open Problems

Known: $O(n*\log^2(n))$-time algorithm for deciding if a polyomino tiles the plane isohedrally.

O(n)-time algorithm?

Enumeration of tilings in O(n*log^2(n) + k) time?

Extend inputs to polygons?

An FPT algorithm for k-isohedral tilings?

Open Problems

Discussed more in the paper on arXiv:

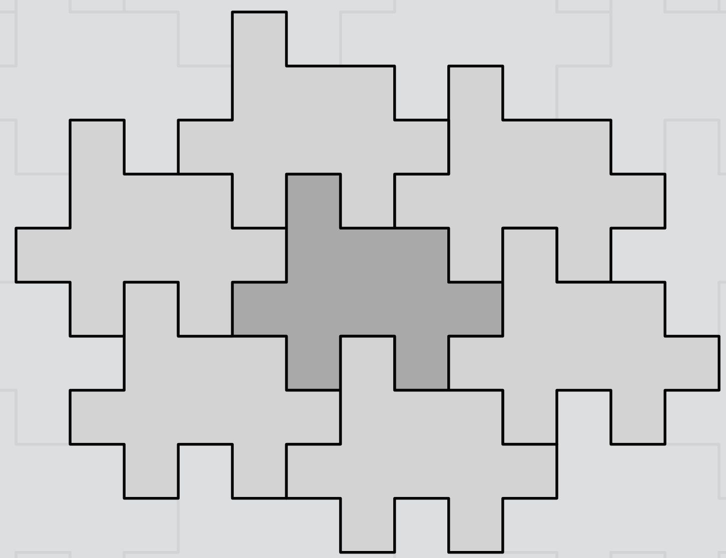
A Quasilinear-Time Algorithm for Tiling the Plane Isohedrally with a Polyomino

Stefan Langerman *   Andrew Winslow *

Abstract

A plane tiling consisting of congruent copies of a shape is *isohedral* provided that for any pair of copies, there exists a symmetry of the tiling mapping one copy to the other. We give a $O(n \log^2 n)$-time algorithm for deciding if a polyomino with n edges can tile the plane isohedrally. This improves on the $O(n^{18})$-time algorithm of Keating and Vince and generalizes recent work by Brlek, Provençal, Fédou, and the second author.

Tiling Isohedrally with a Polyomino



Andrew Winslow
(joint work with Stefan Langerman)

ULB
UNIVERSITE
LIBRE
DE BRUXELLES