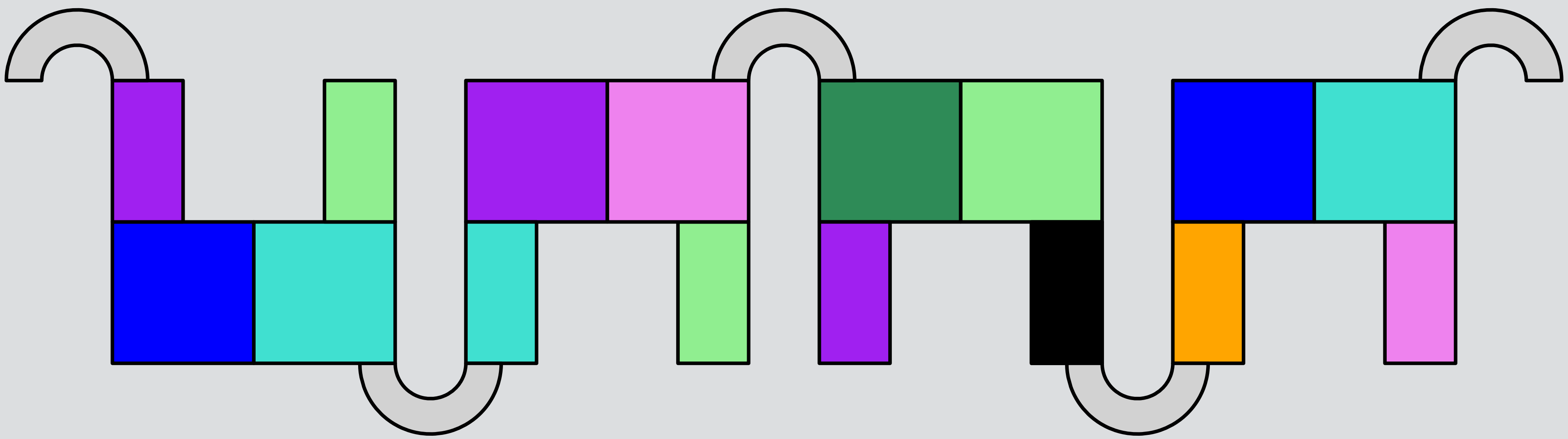
The Limits of a Simple Model of Active Self-Assembly



Andrew Winslow

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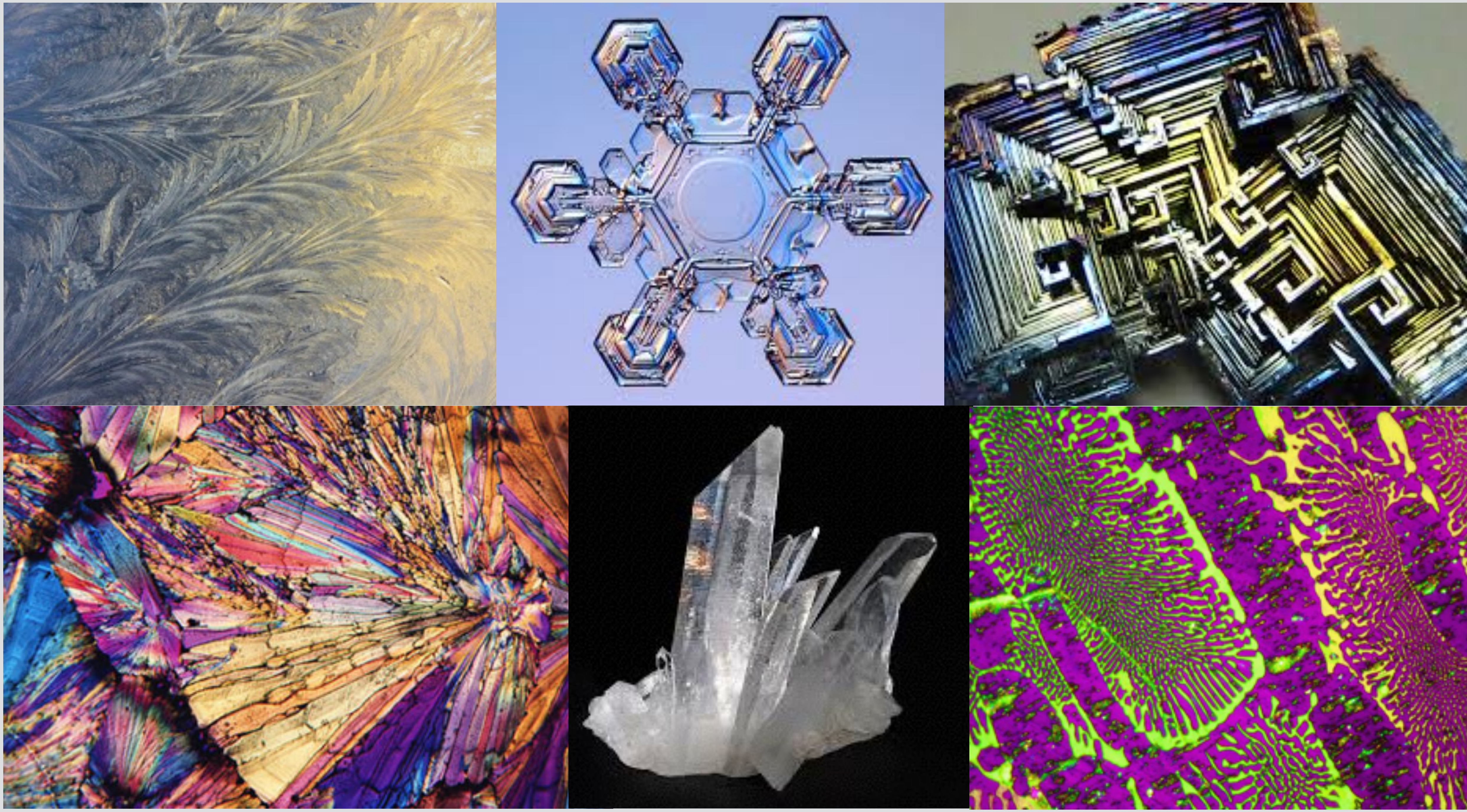
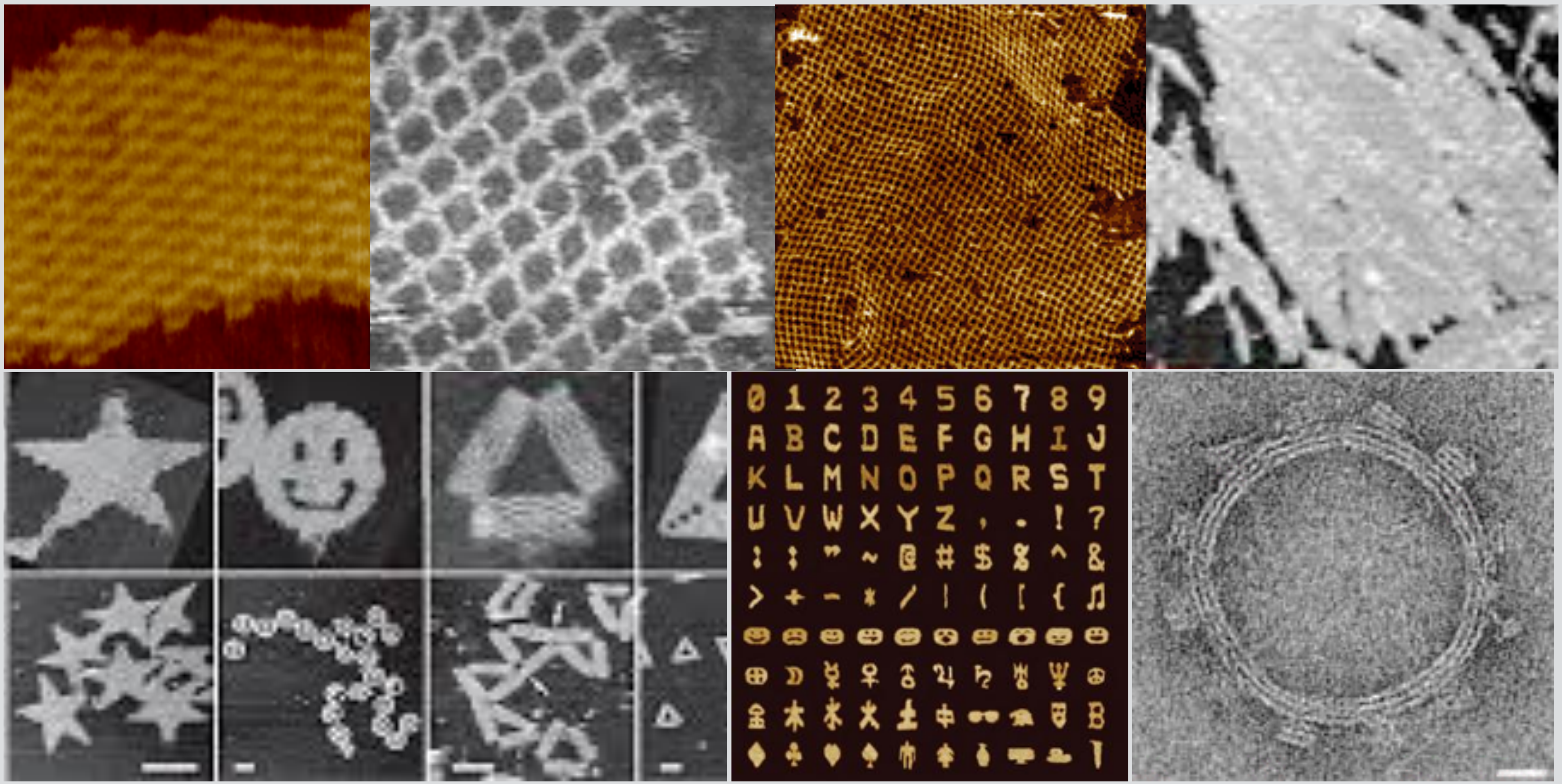
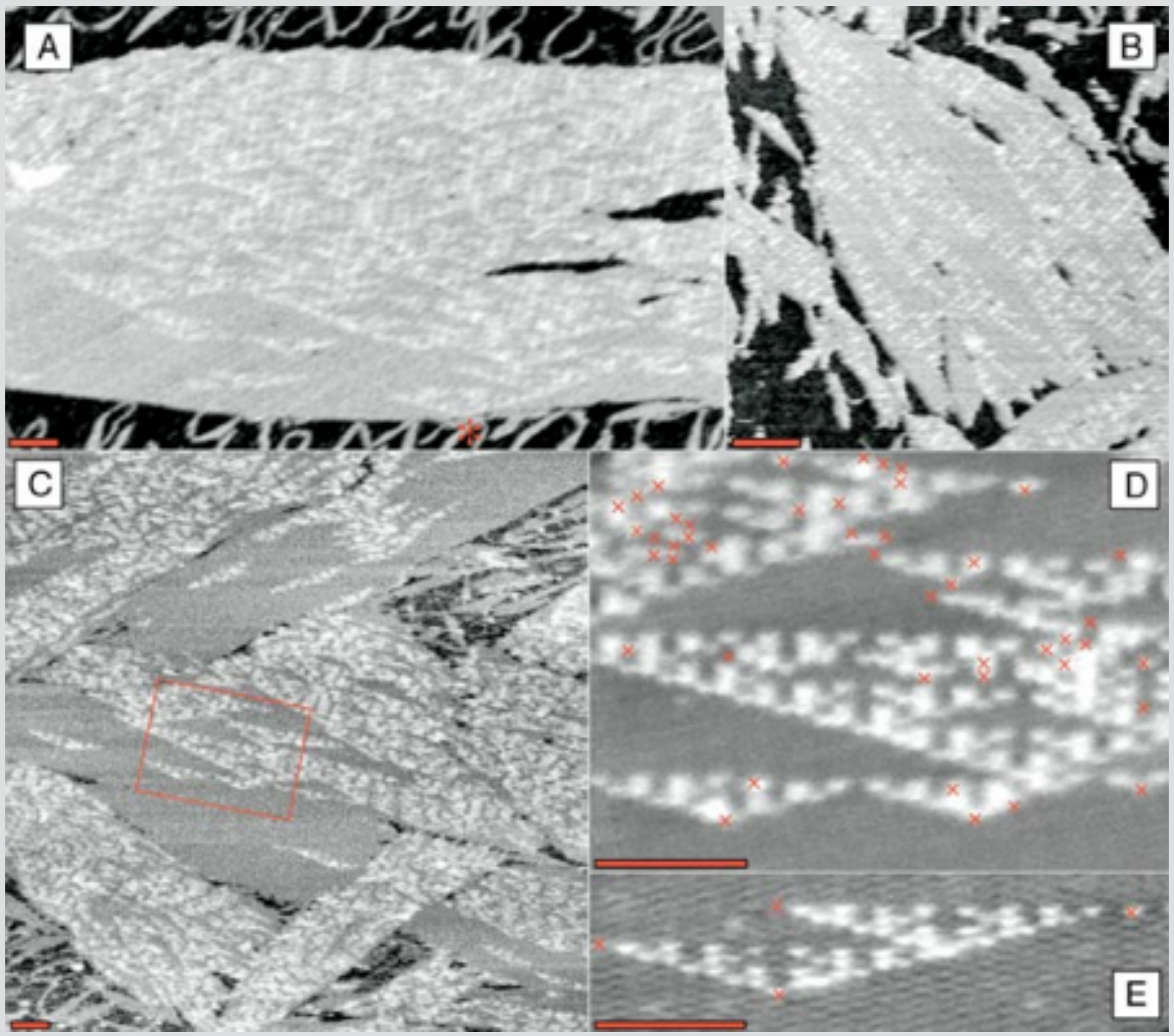
Natural Nanoscale Self-Assembly



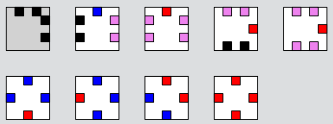
Synthetic Nanoscale Self-Assembly



Implementation (in DNA)



Theoretical model (aTAM)



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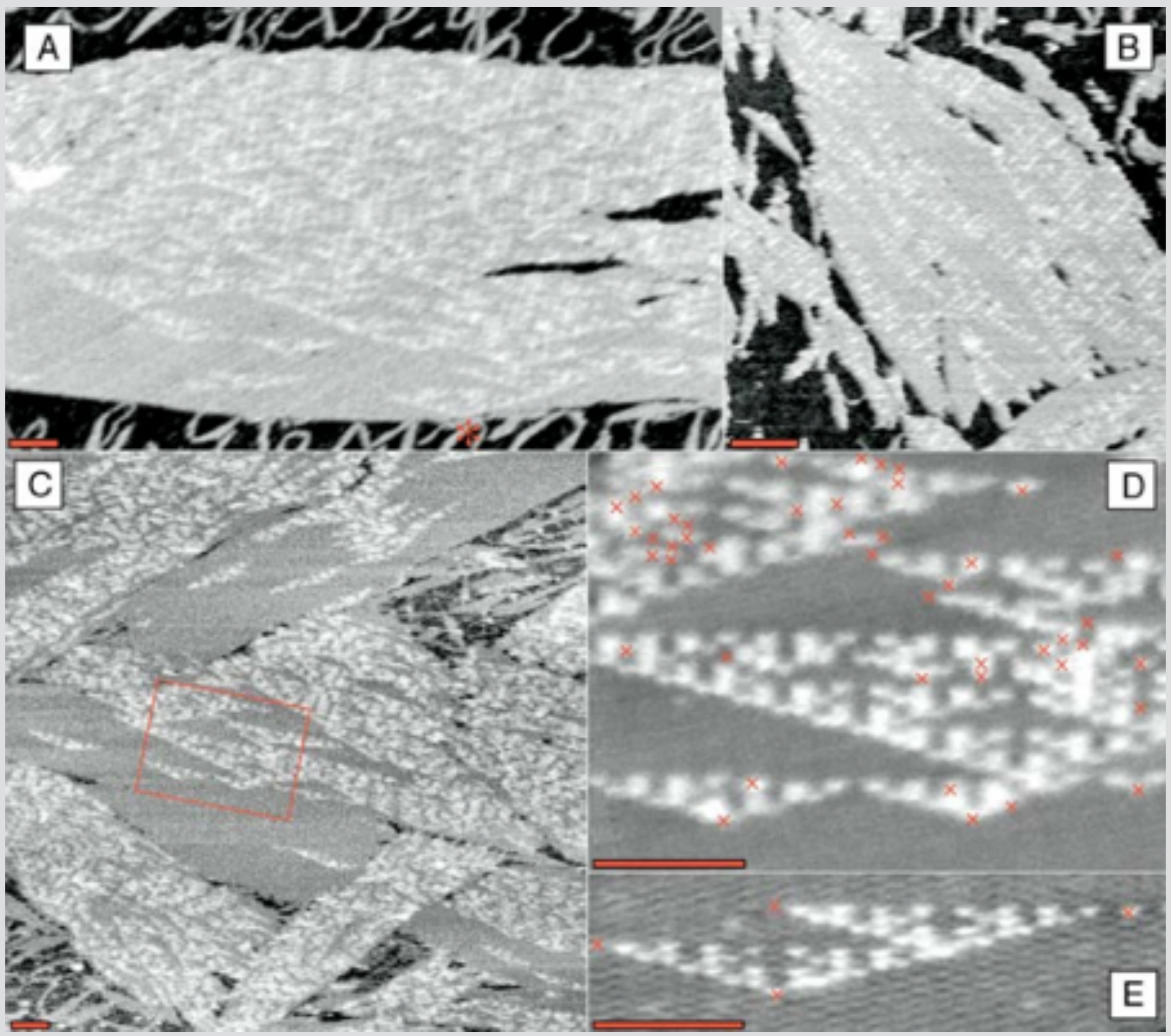

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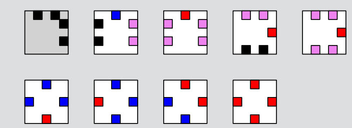

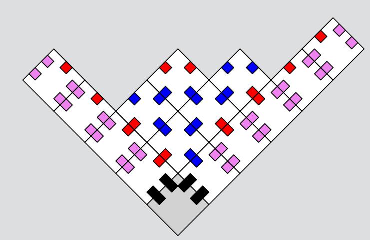

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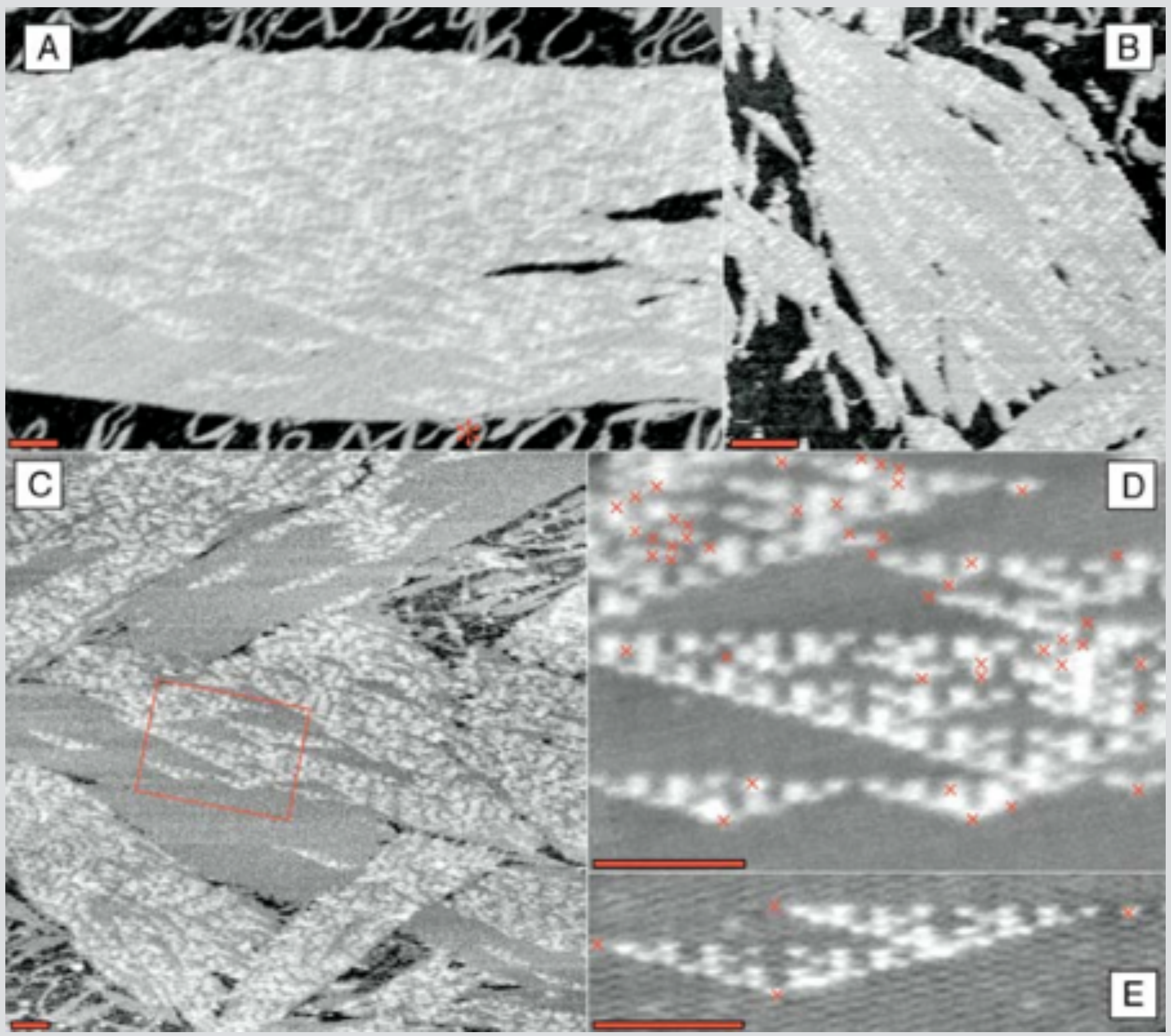

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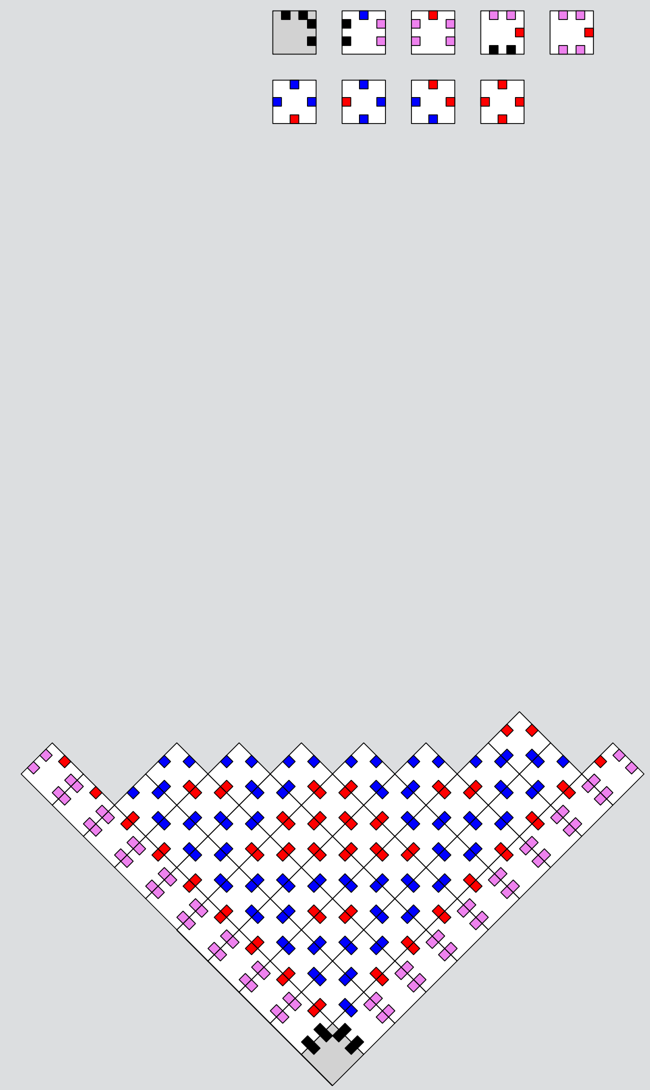

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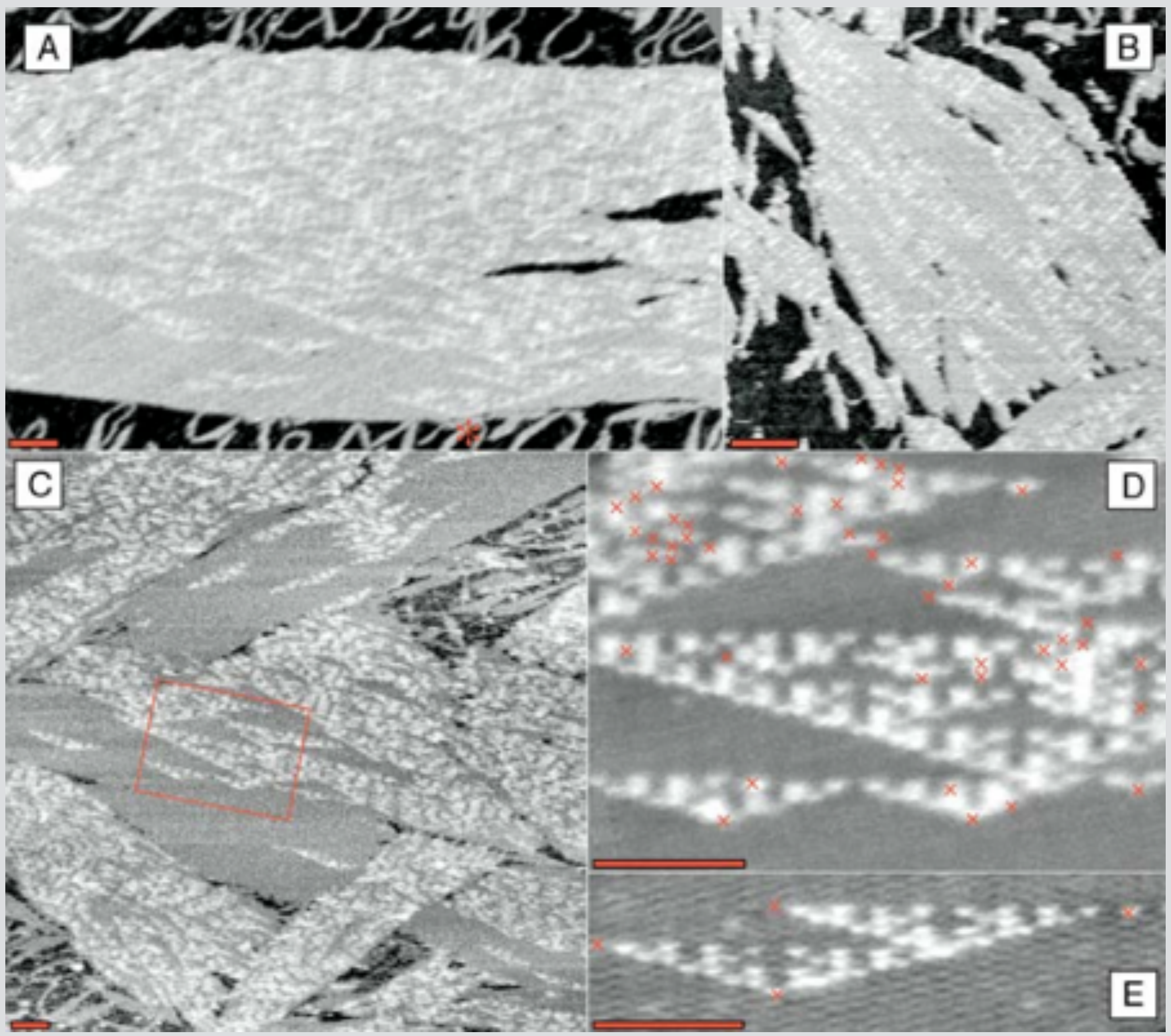

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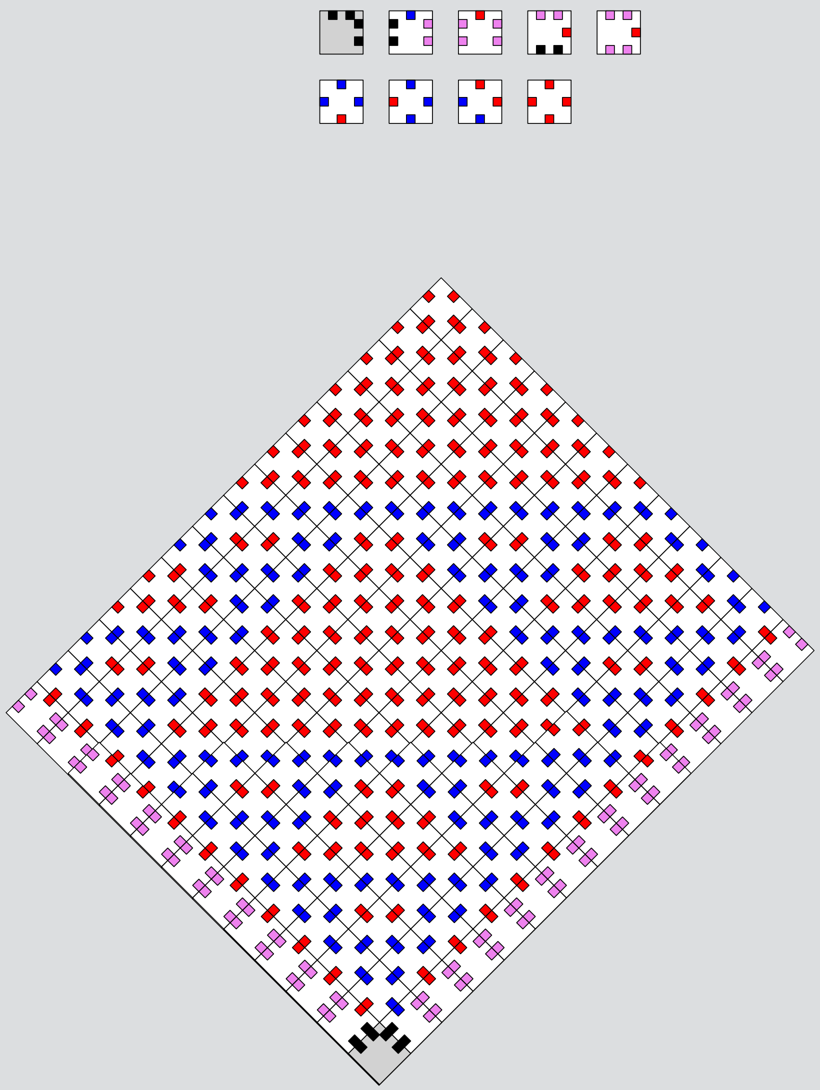

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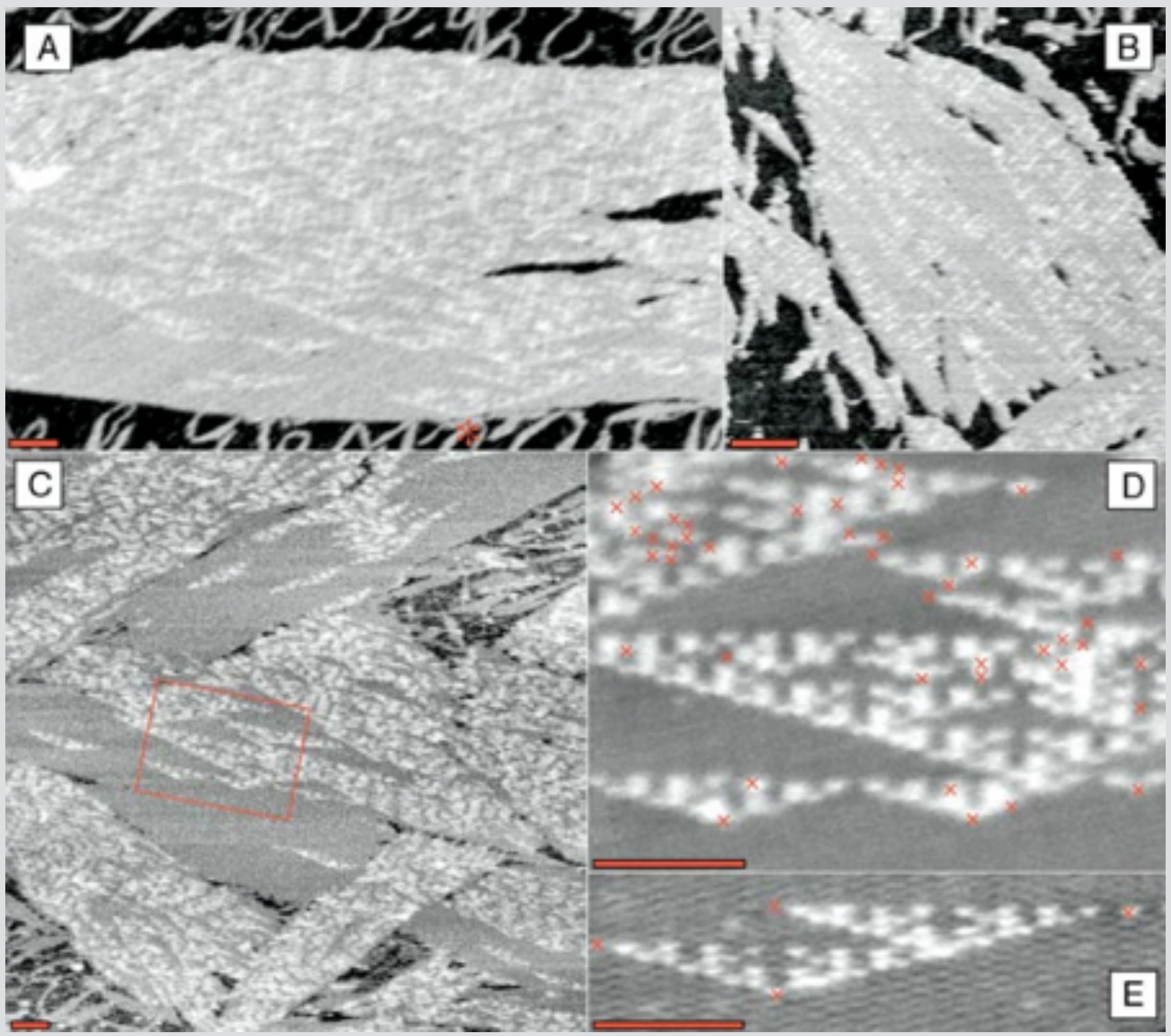

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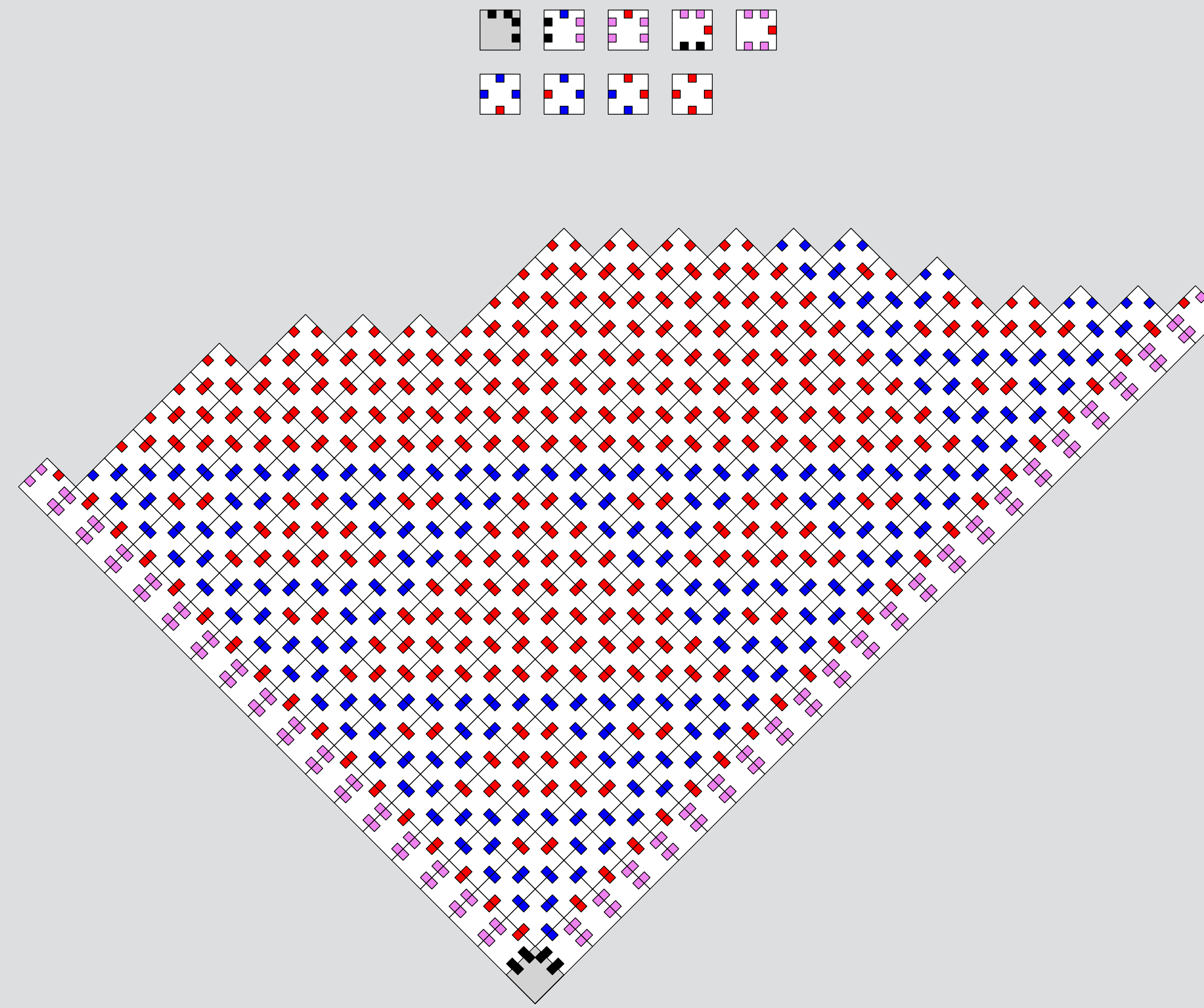
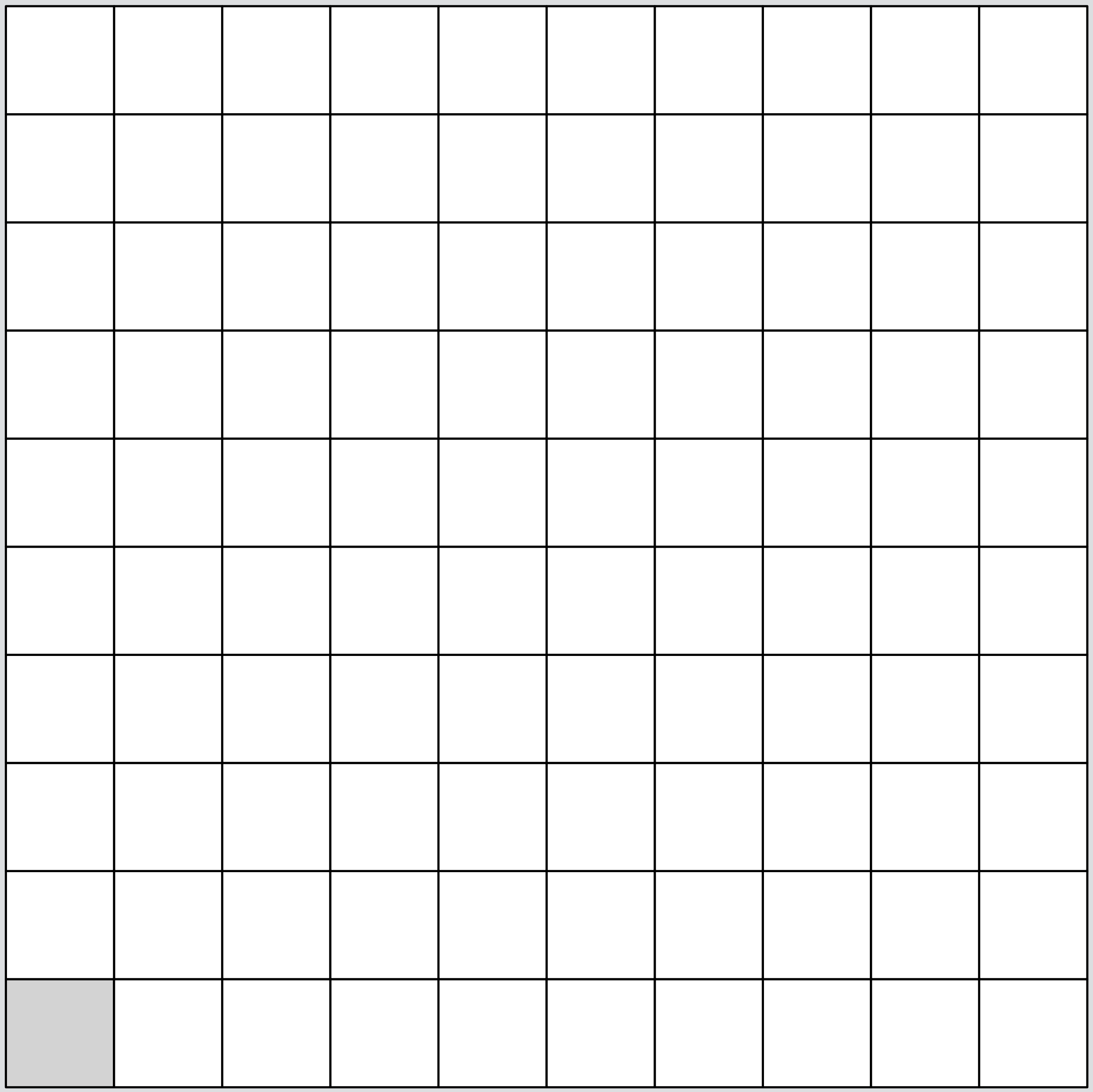
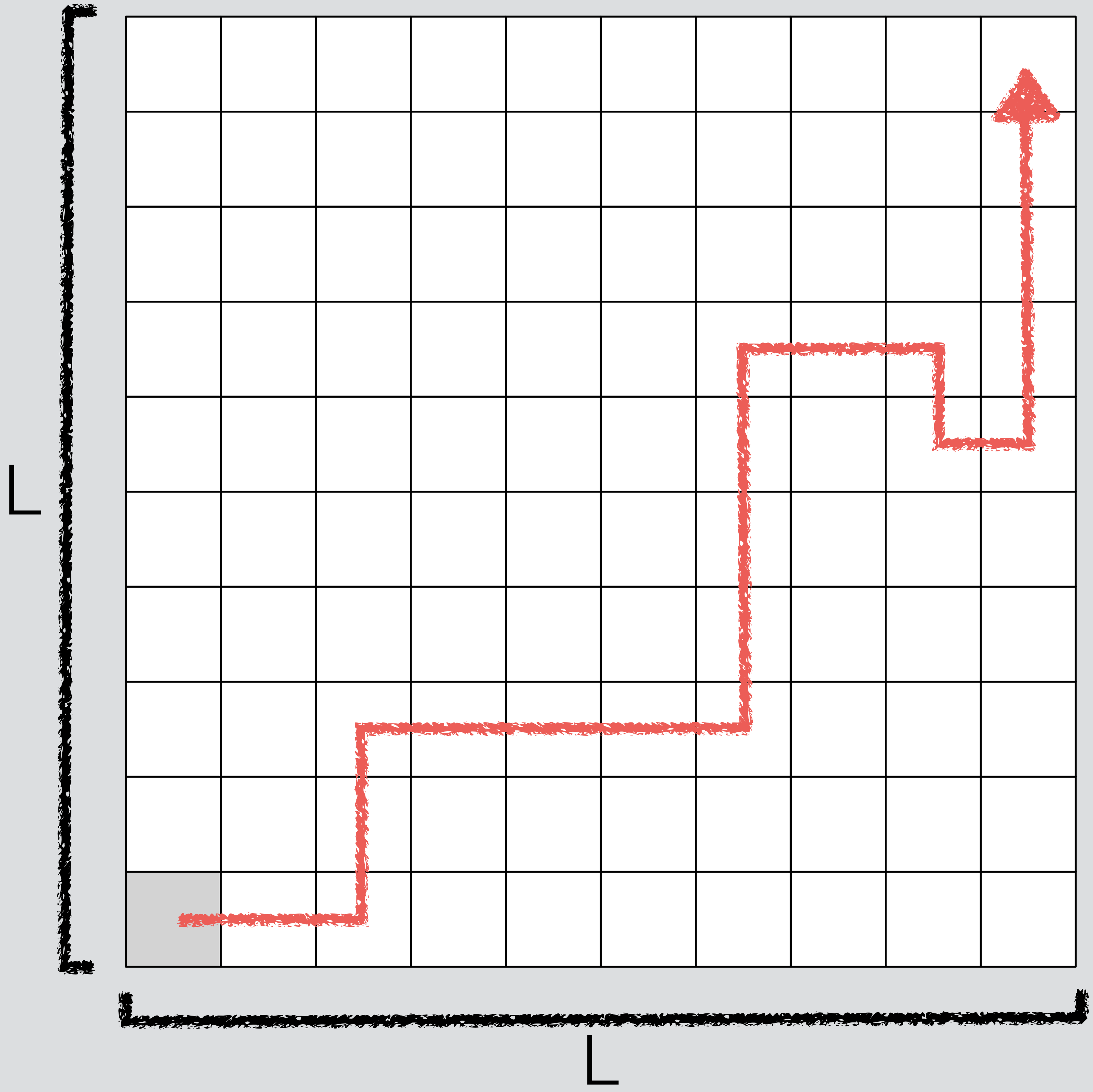


A growth rate limitation

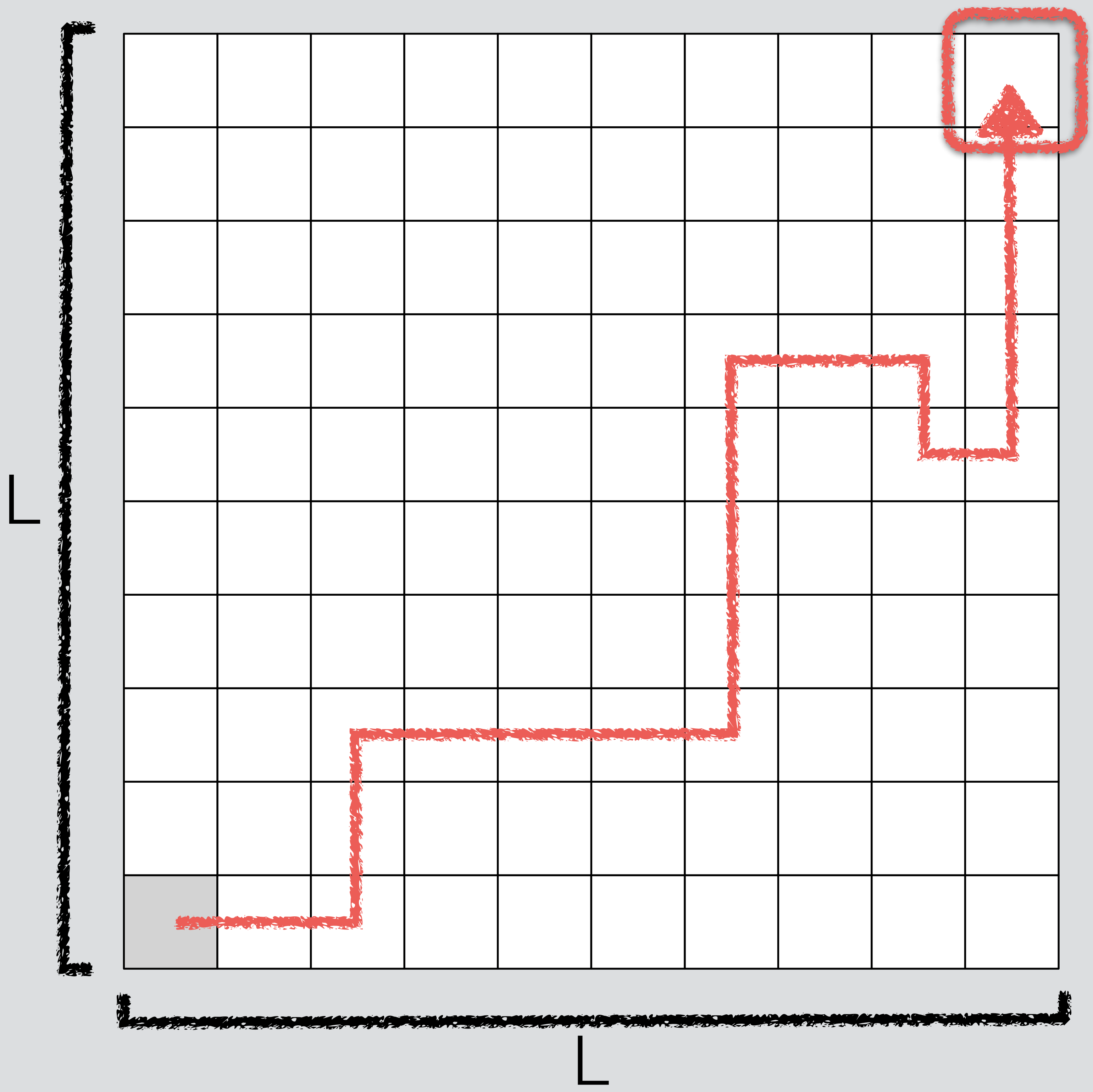


A growth rate limitation



path length ≥ L

A growth rate limitation



path length ≥ L

at least L tile prior tile placements

Growth rates

- Takes Ω(diameter) = $\Omega(n^{0.5})$ expected time for n particles to assemble on a square lattice.

- So growth rate is $O(n^2)$.

- Similar polynomial bounds for other lattices.

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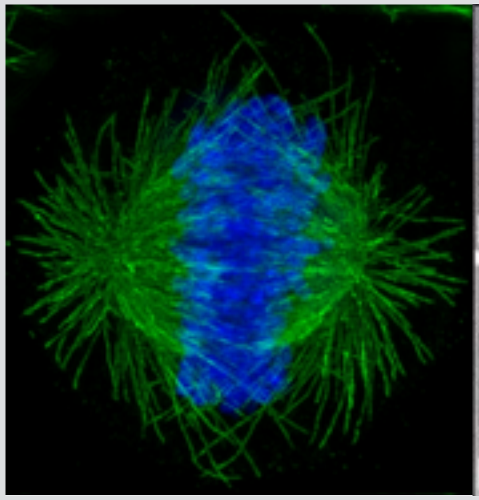
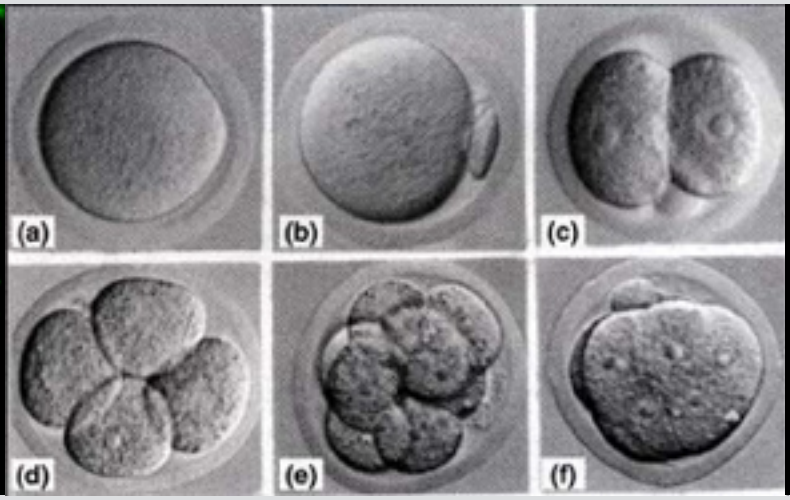
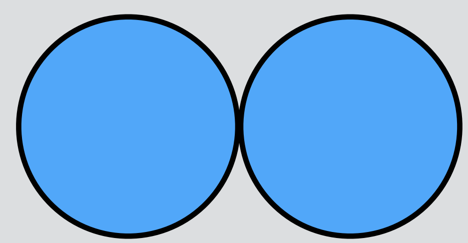
- Takes $\Omega(\text{diameter}) = \Omega(n^{0.5})$ expected time for n particles to assemble on a square lattice.
- So growth rate is $O(n^2)$.
- Similar polynomial bounds for other lattices.
- But not all nanoscale growth is polynomial!

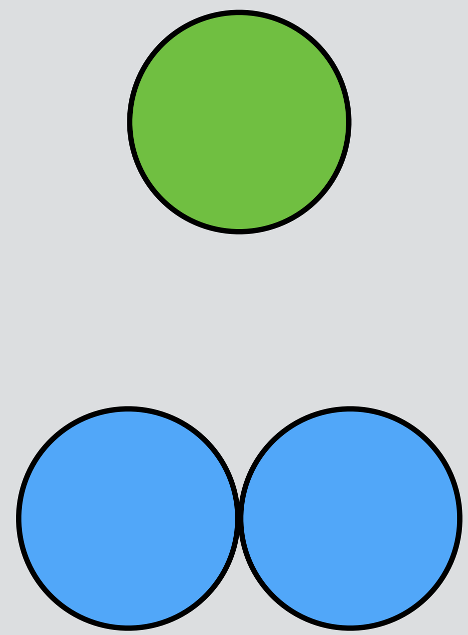


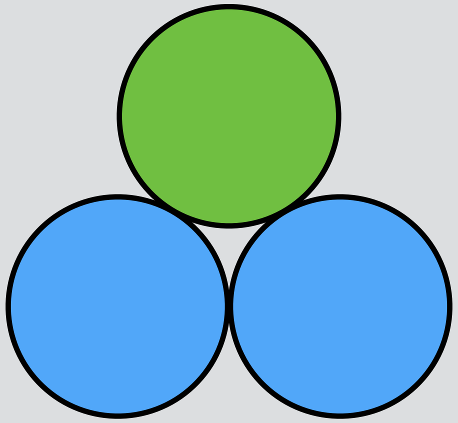


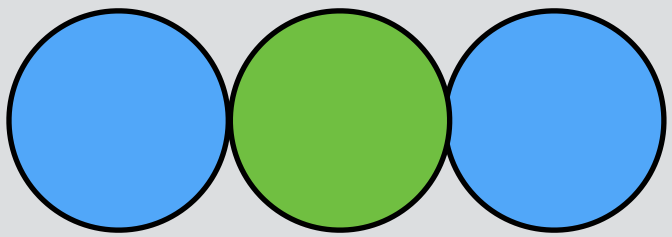
Passive vs. Active Self-Assembly

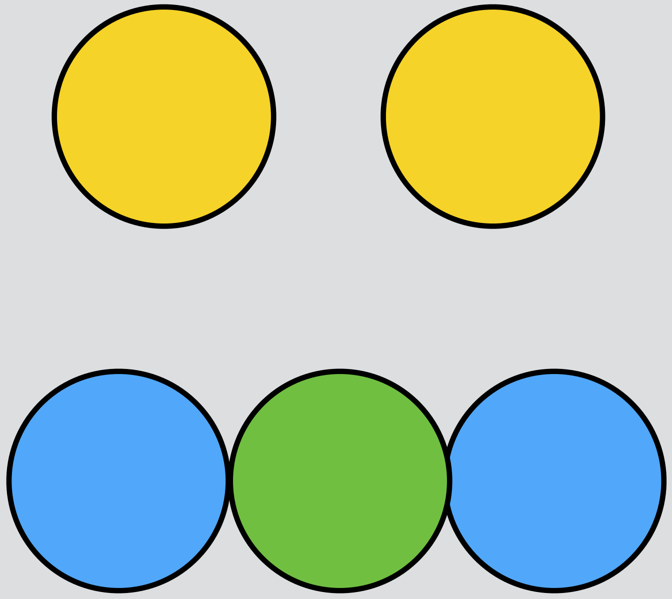
- Most current DNA self-assembly uses crystal-like **passive** growth: bonds and geometry do not change.
- Some natural systems use **active** growth: bonds and geometry change.
- Active growth enables exponential growth rates.

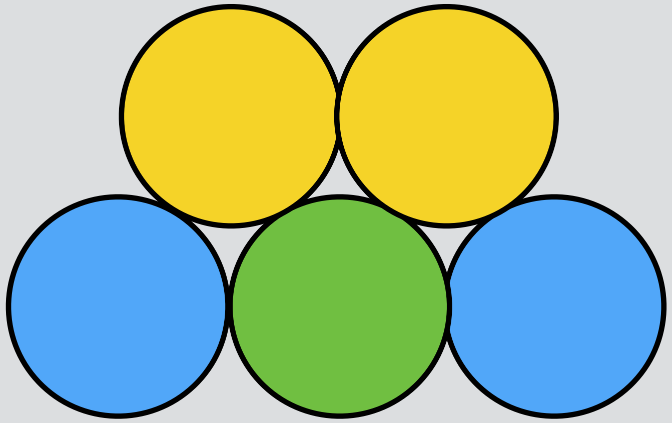


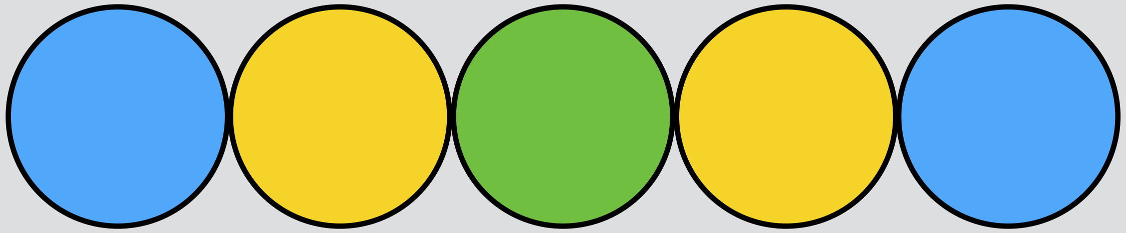


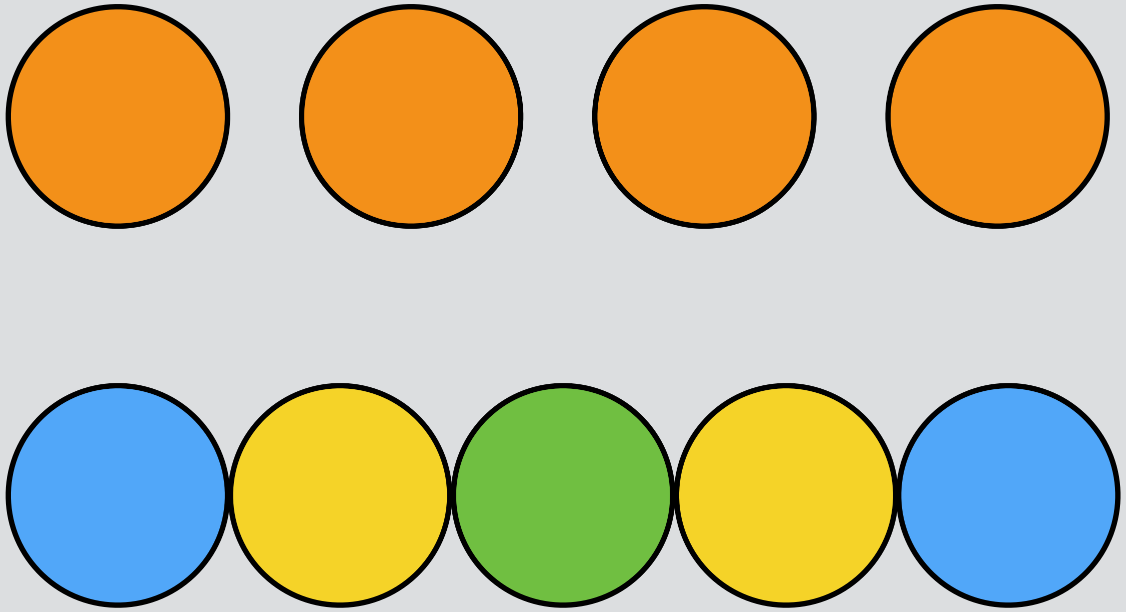


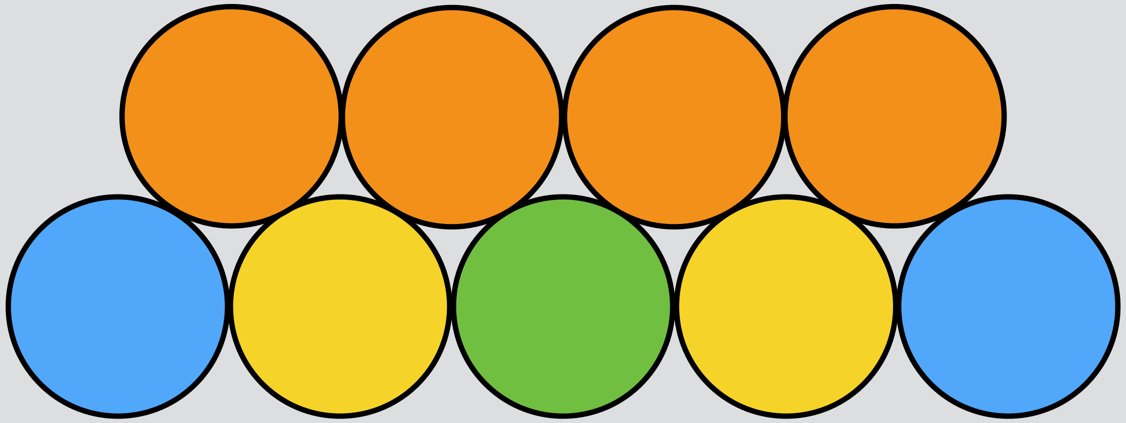


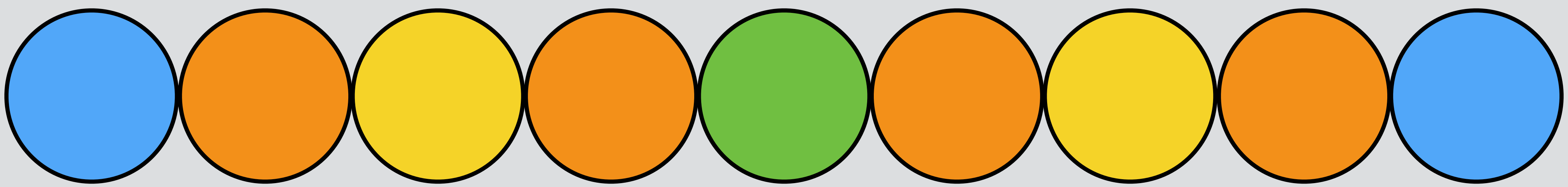


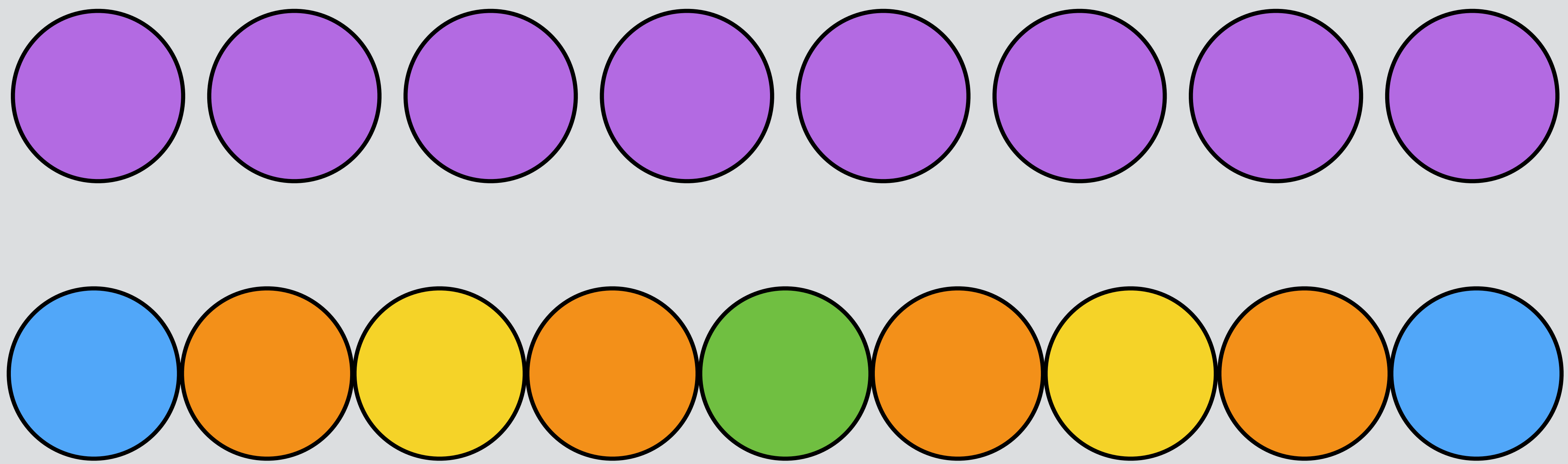


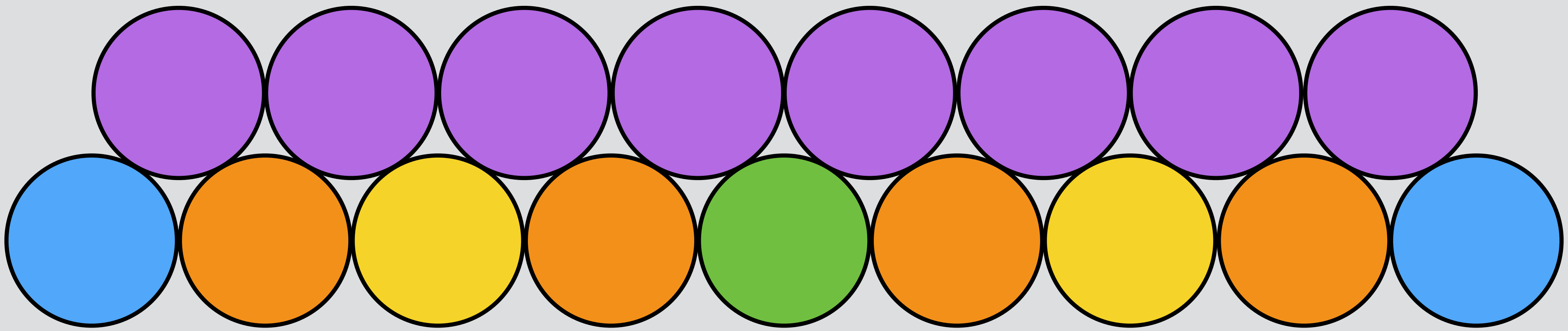


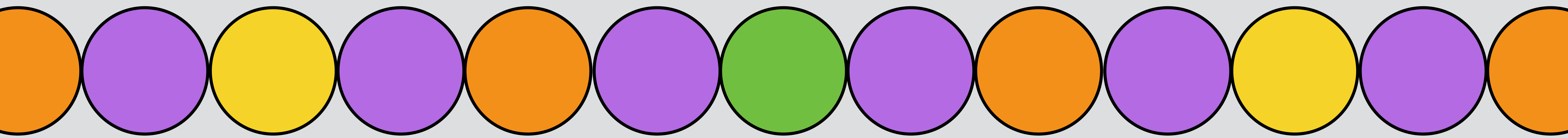




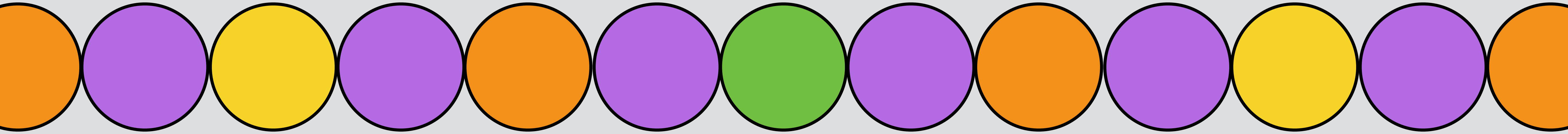






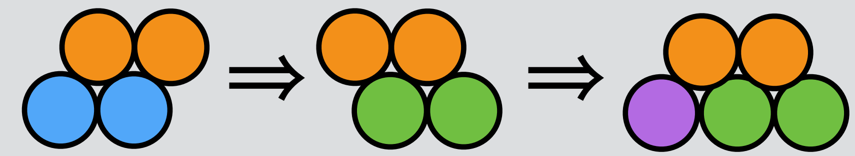


Exponential growth!

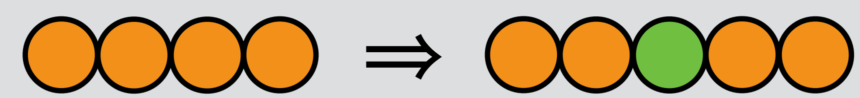


Active self-assembly models

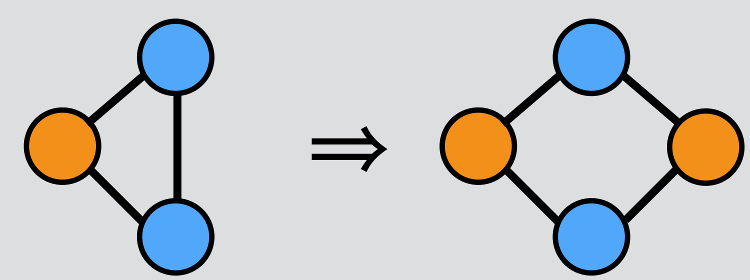
Nubots [Woods et al. ITCS 2012]:
2D, flexible and rigid bonds, stateful particles.



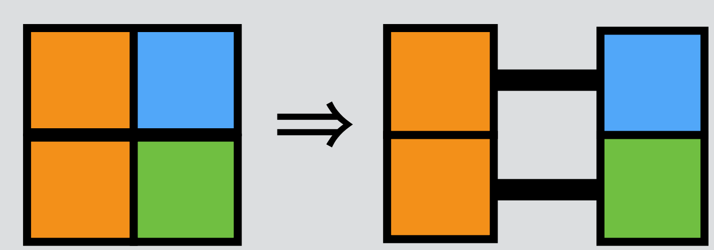
Insertion systems [Dabby, Chen SODA 2013]:
1D, fixed shape, stateless particles.



Graph grammars [Klavins et al. ICRA 2004]:
Geometry-less, stateless particles.



Crystalline robots [Rus, Vona ICRA 1999]:
3D, stateful particles, global communication.



Insertion systems

- Introduced by [Dabby, Chen 2013].
- A model of active self-assembly.
    - Implementable in DNA.
    - Capable of exponential growth.
    - Grows a linear structure by insertion of particles.

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- A model of active self-assembly.
    - Implementable in DNA.
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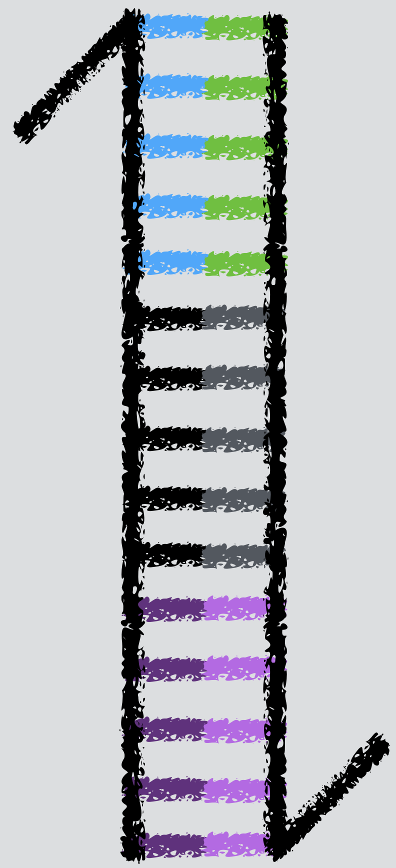
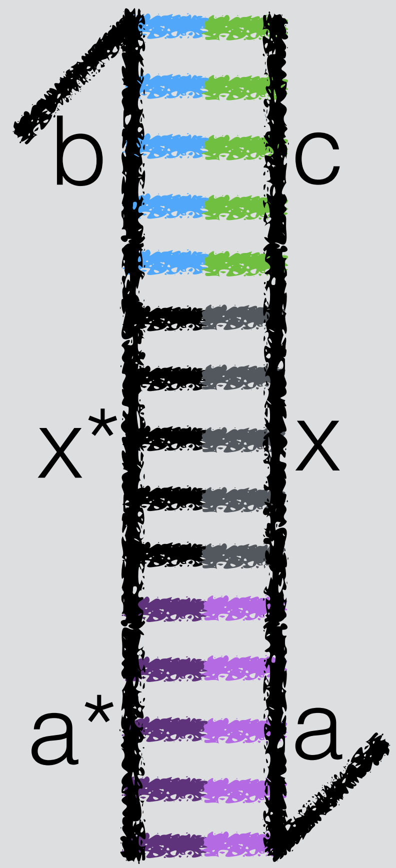
    - Grows a linear structure by insertion of particles.
- Our work: bound capabilities of insertion systems.

DNA insertions



DNA insertions



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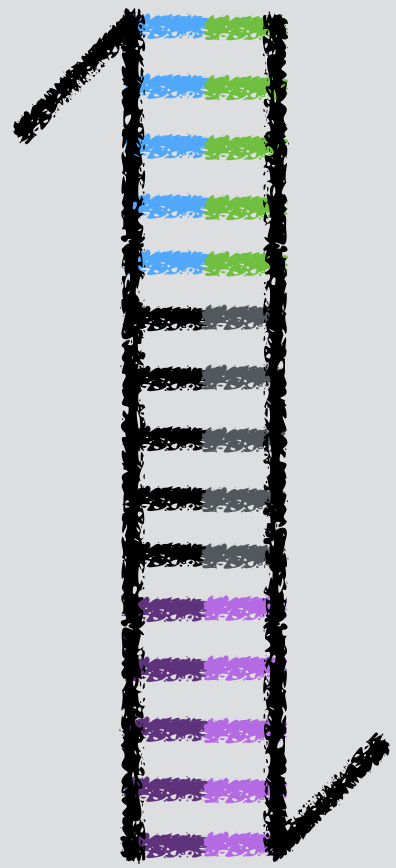

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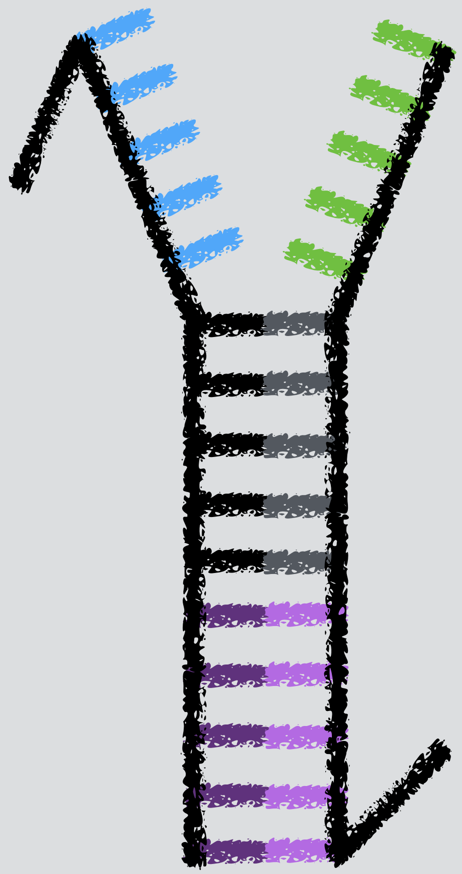

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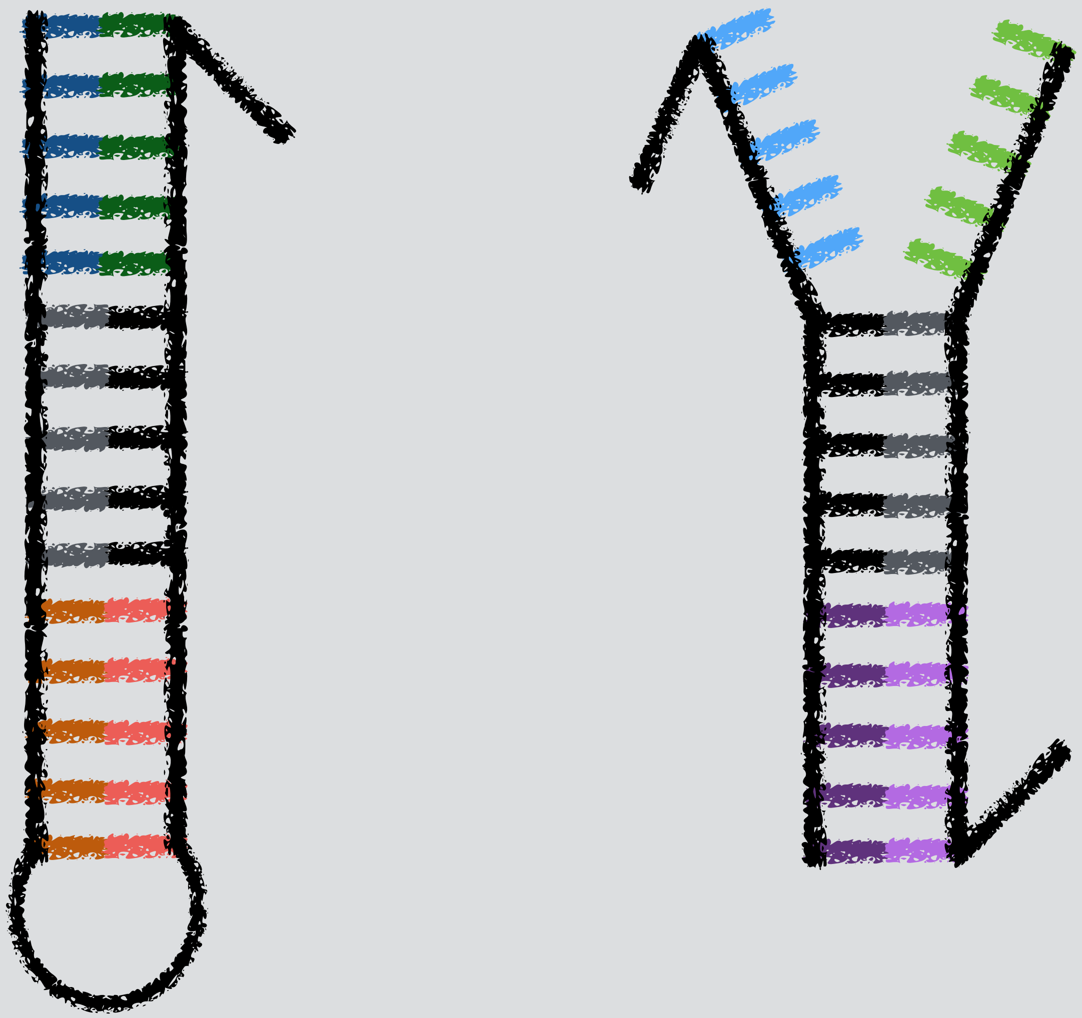

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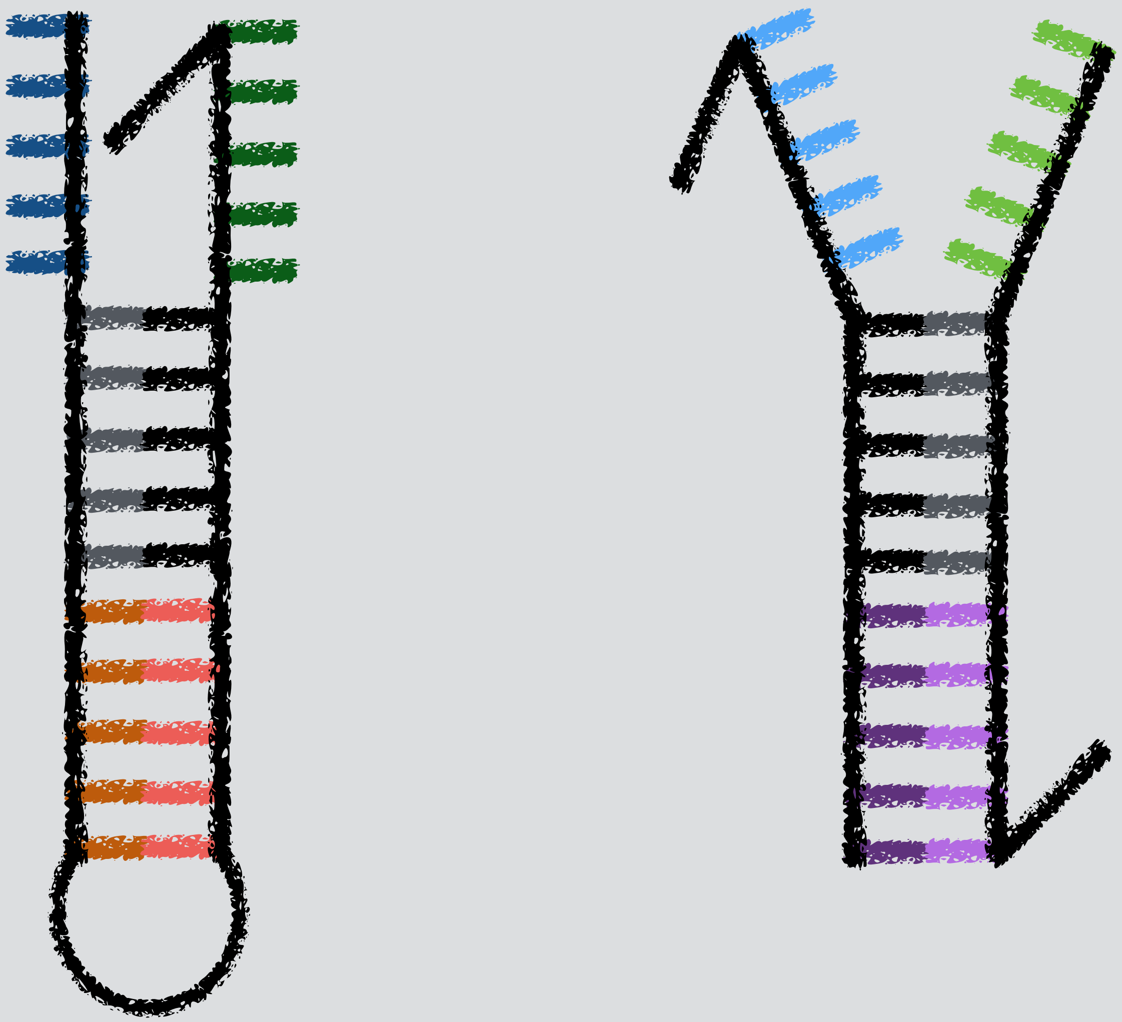

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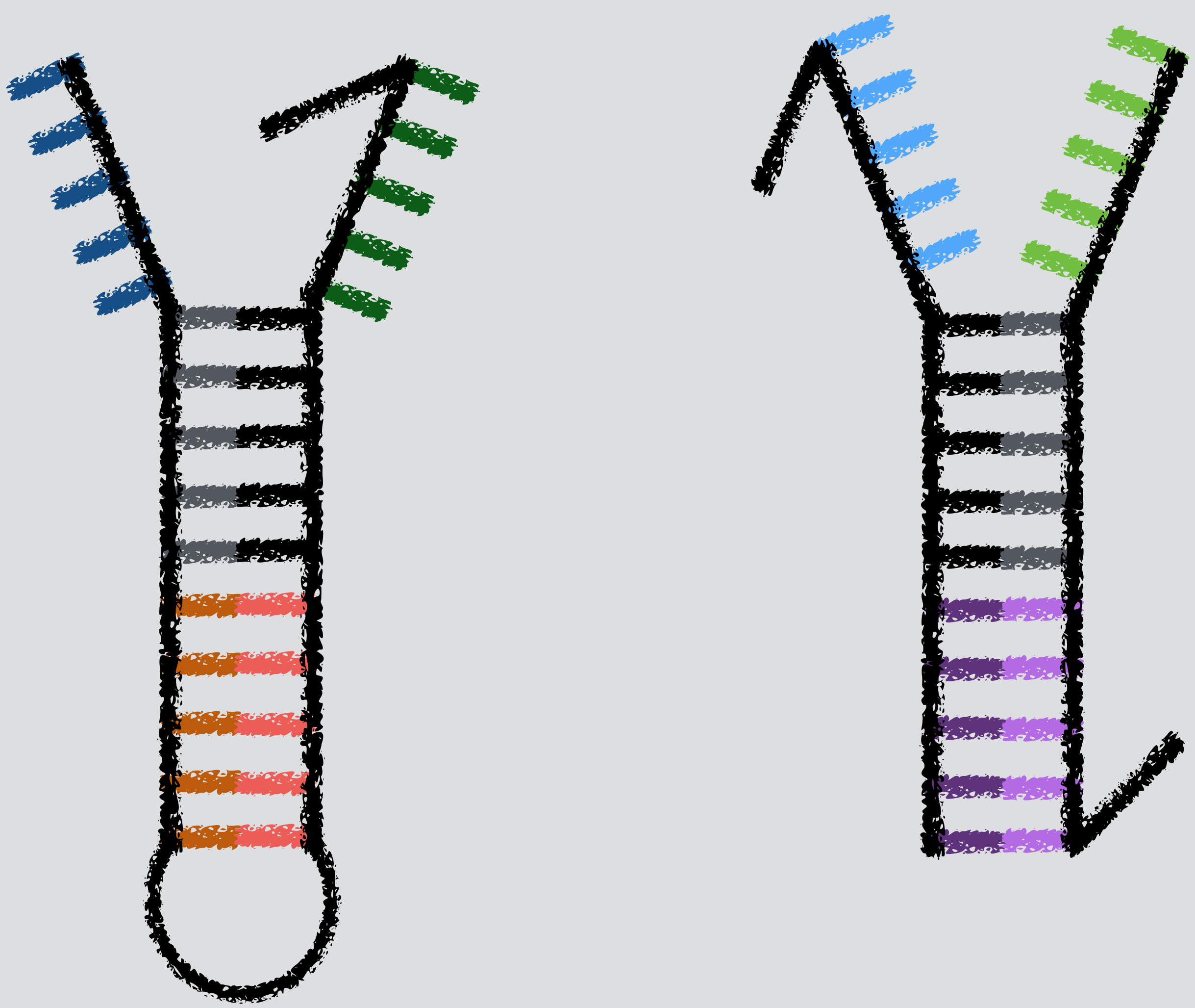

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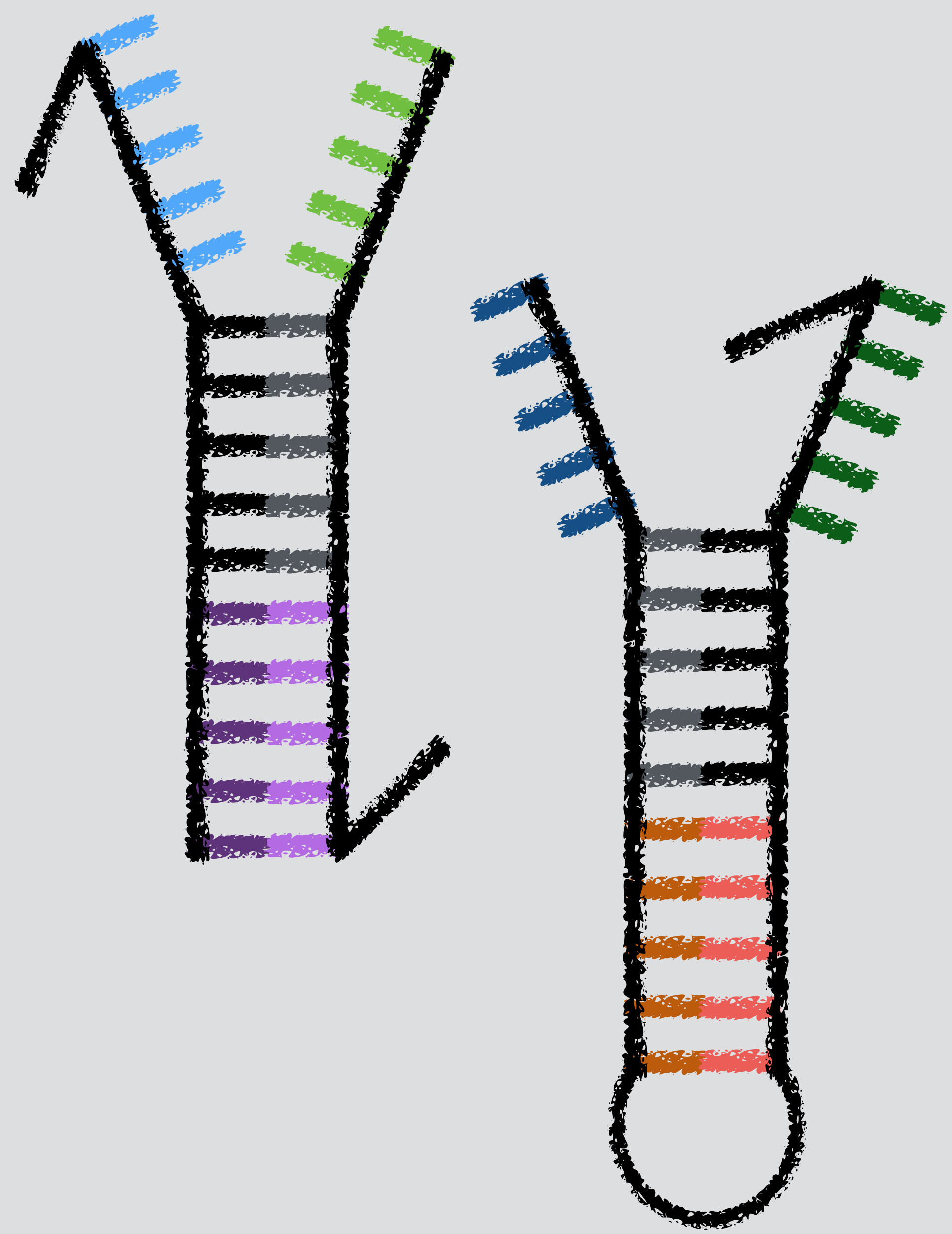

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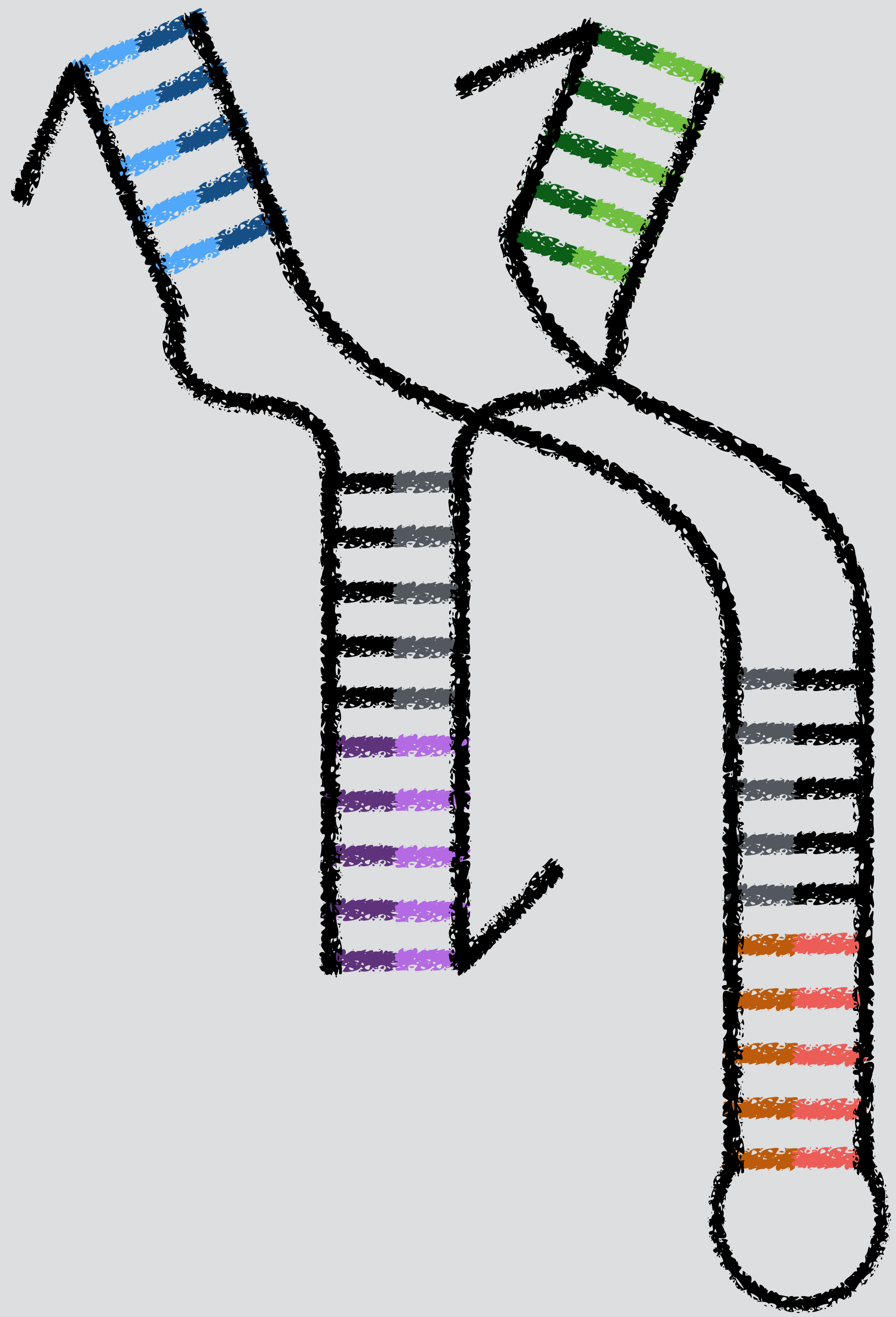

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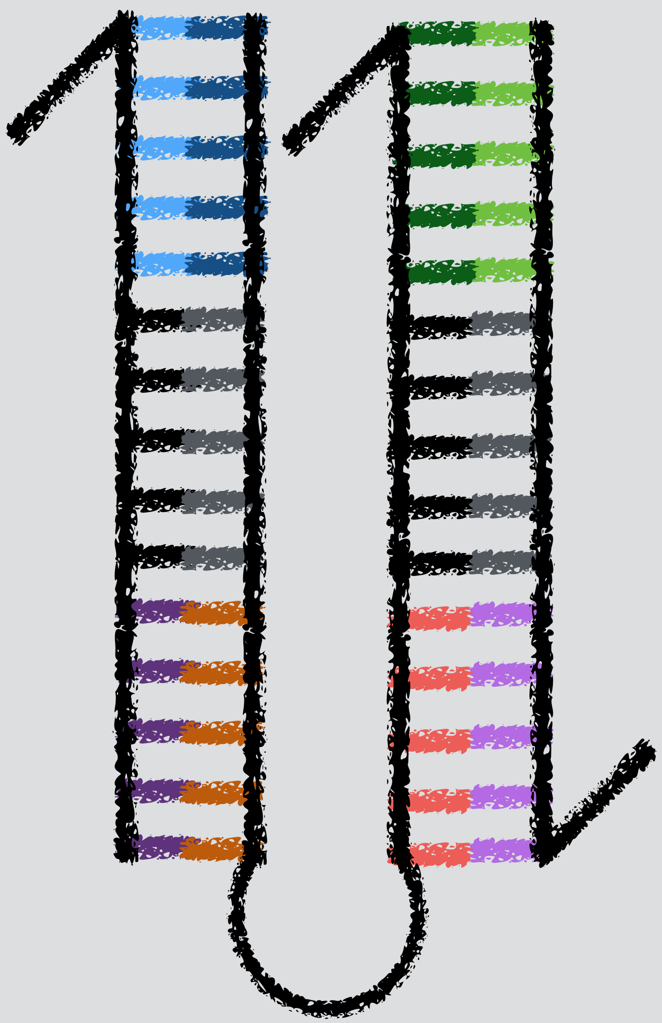

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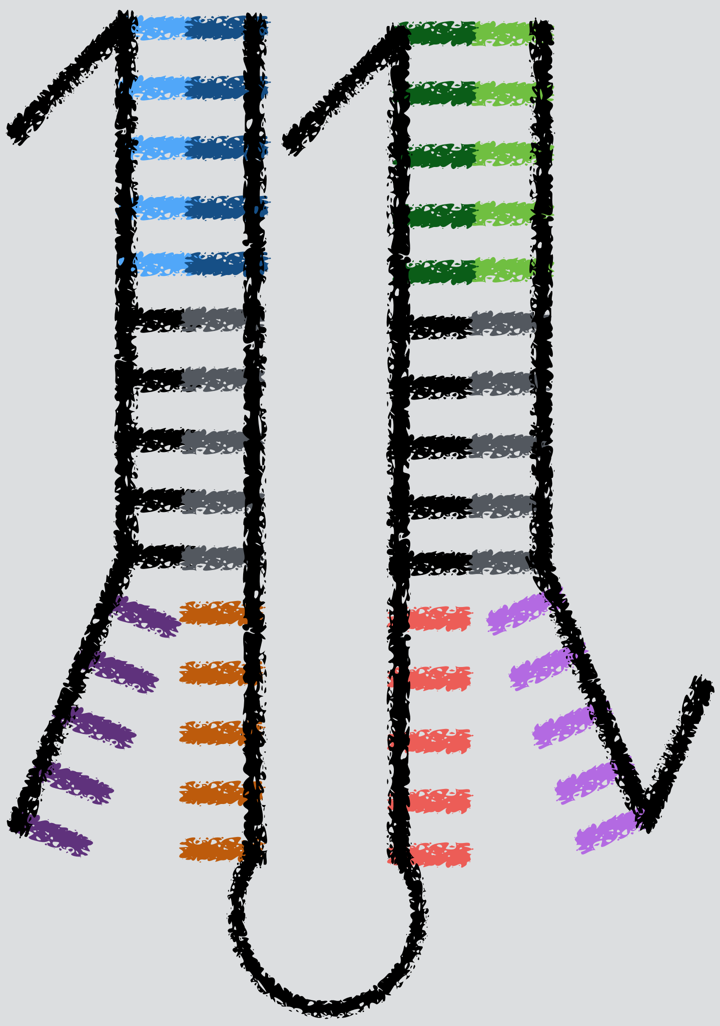

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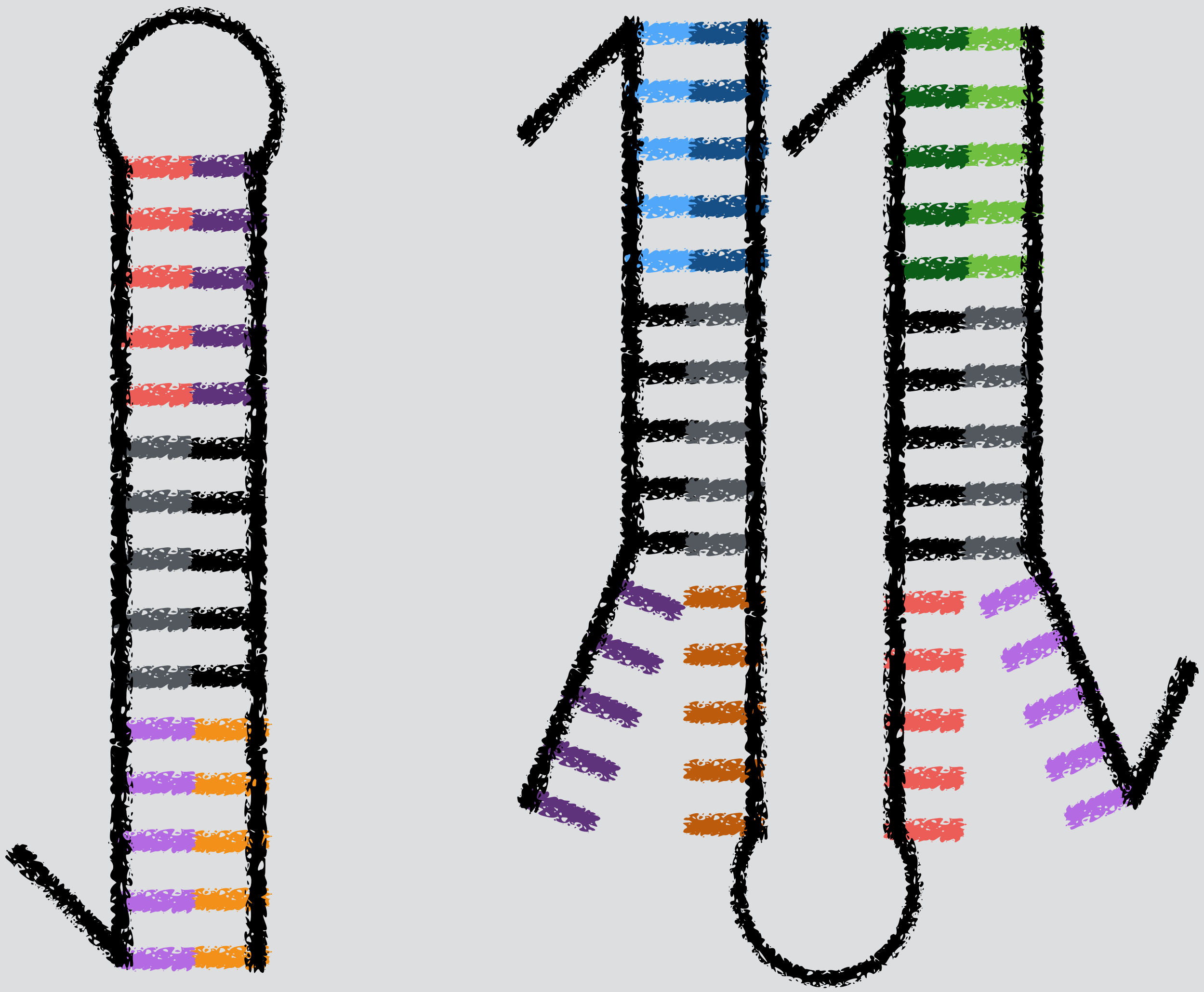

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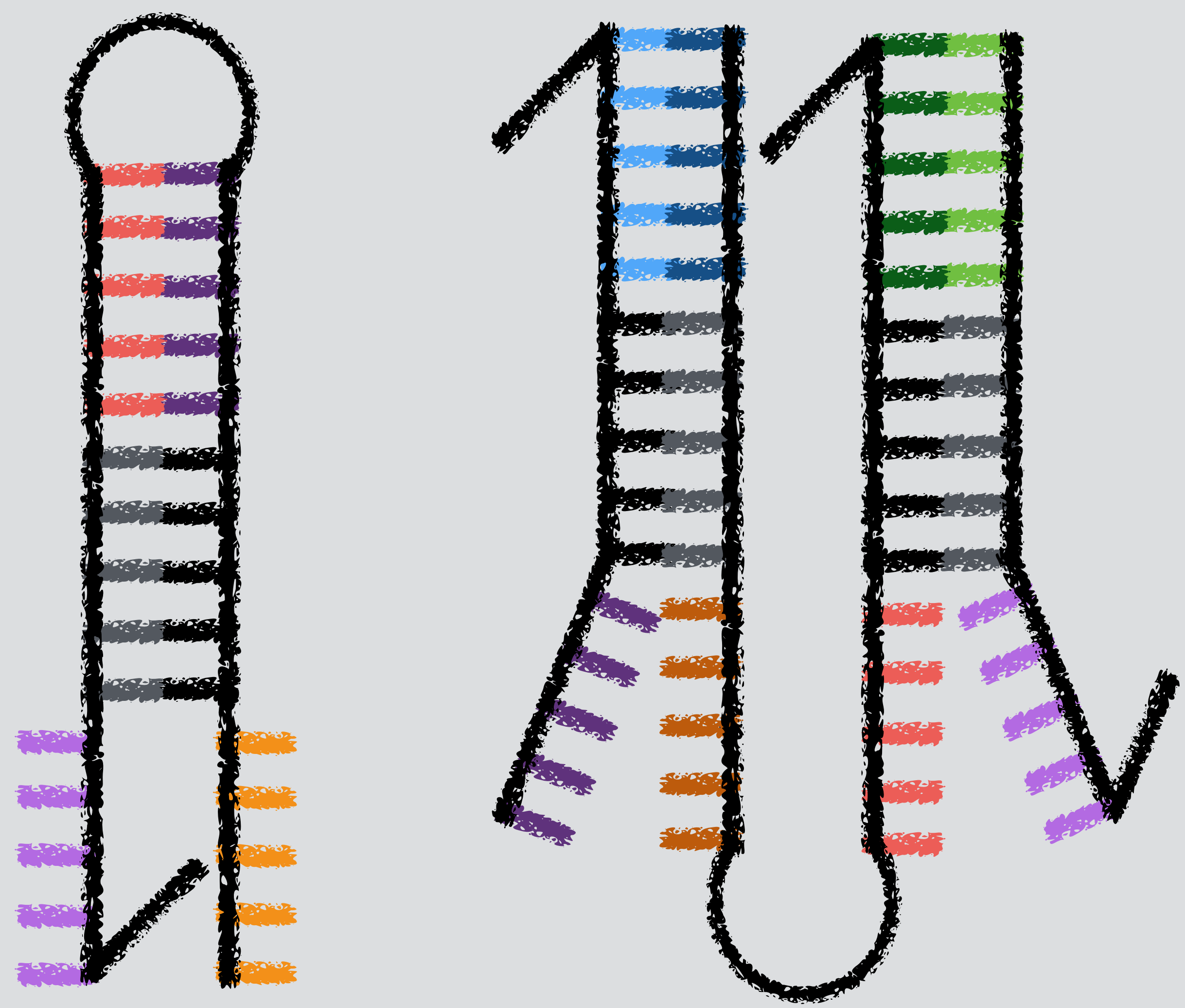
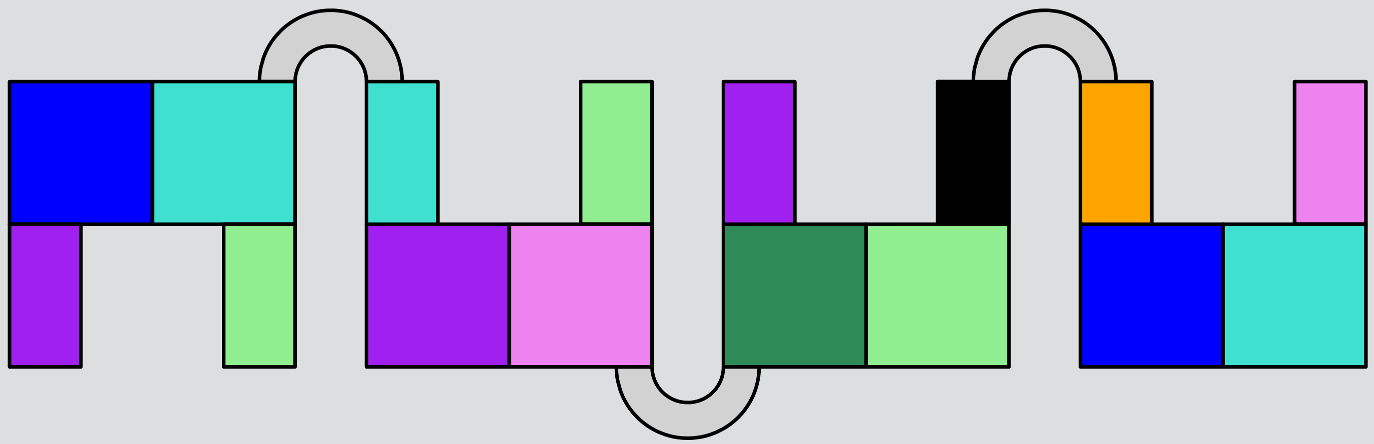


Insertion systems



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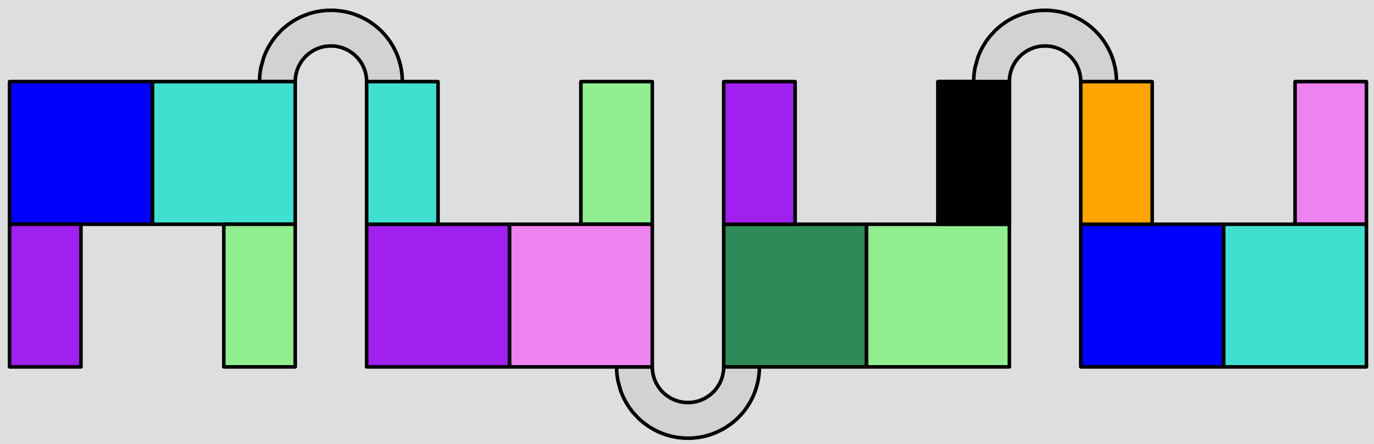

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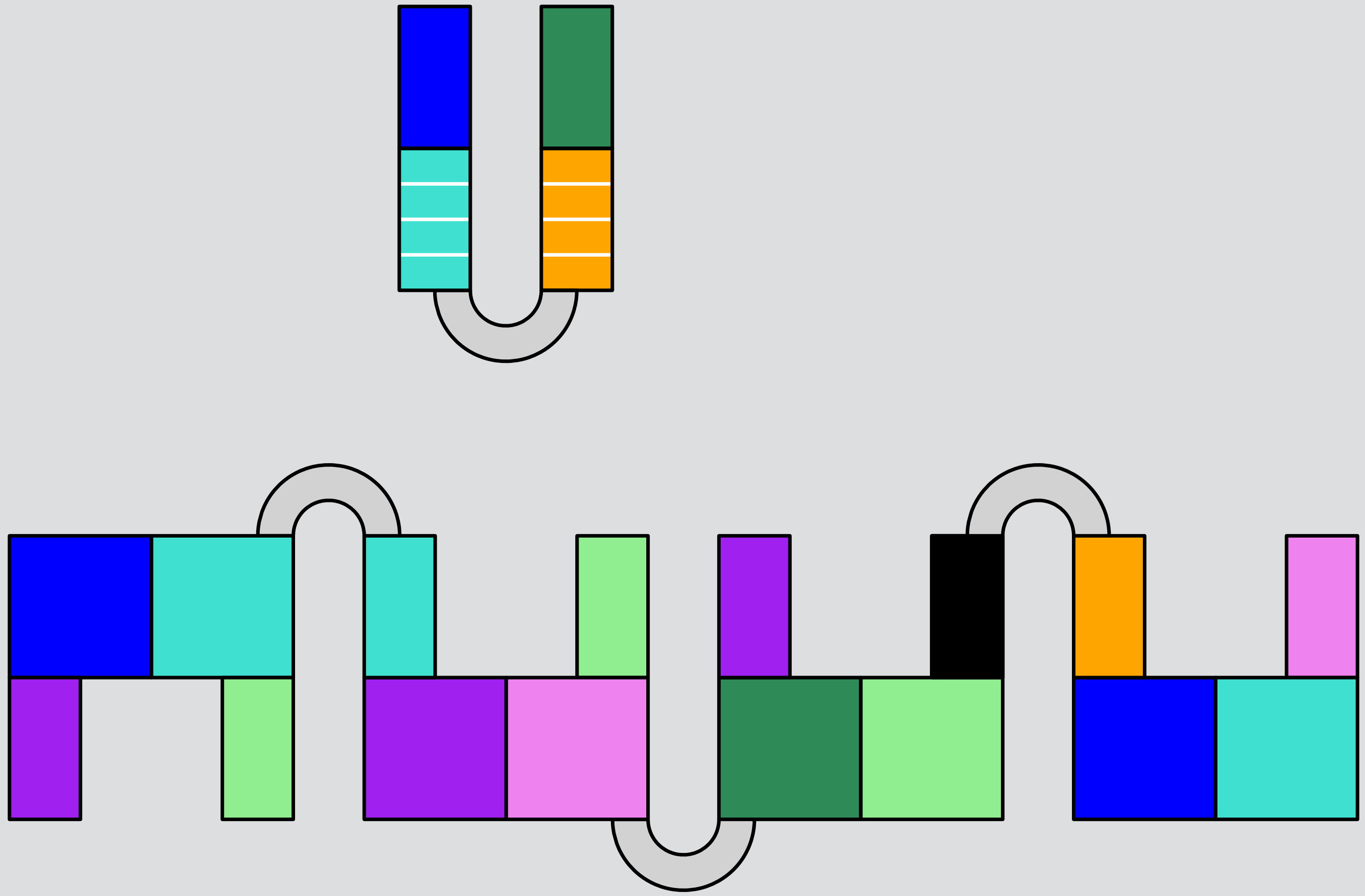

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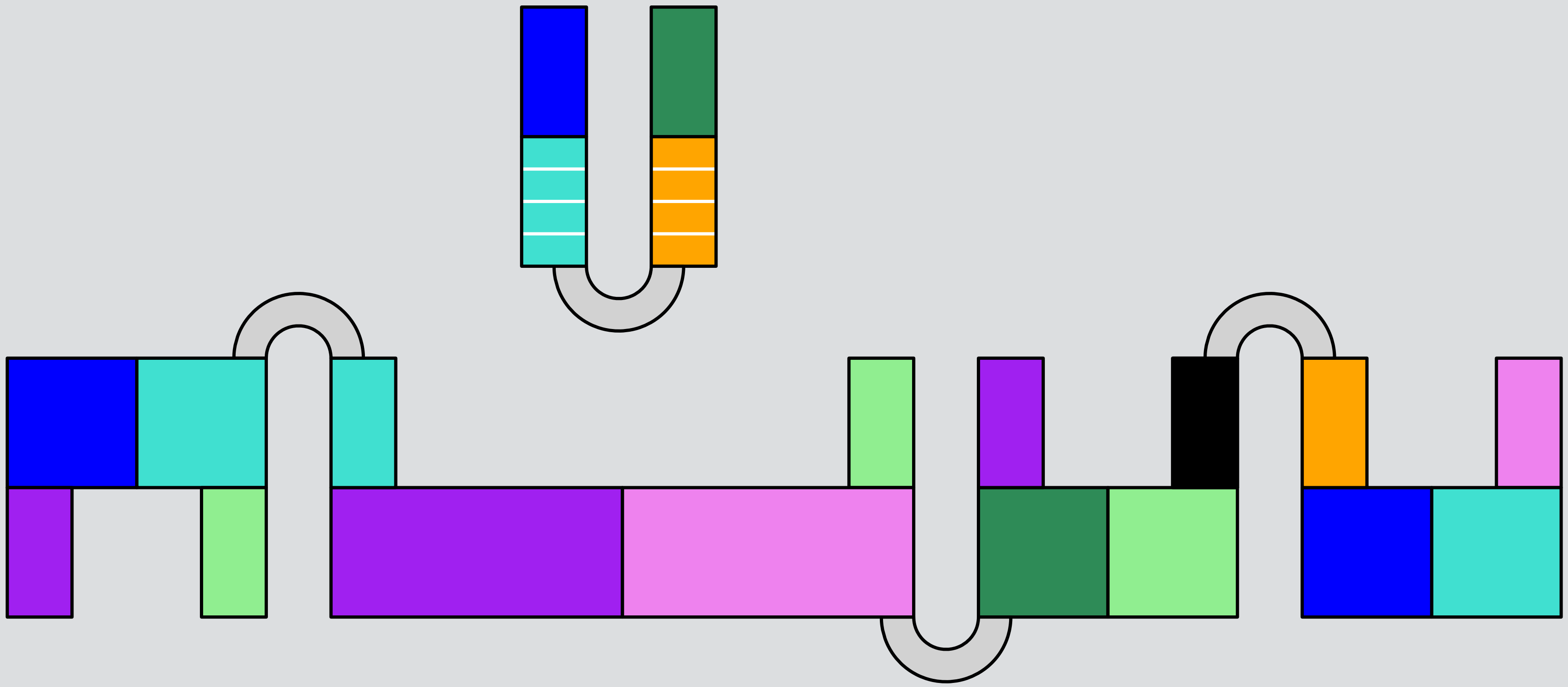

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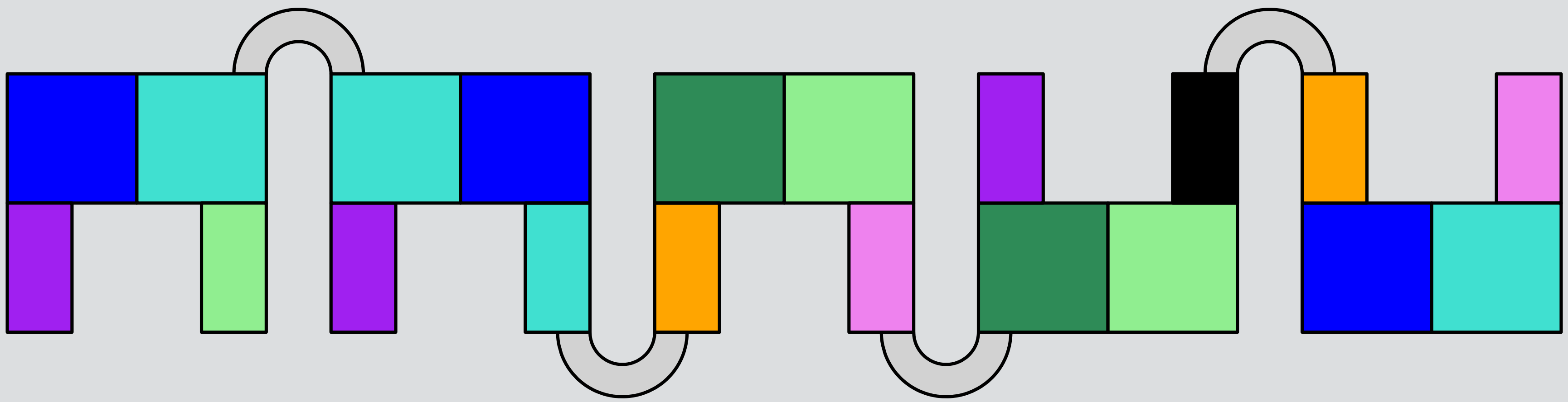
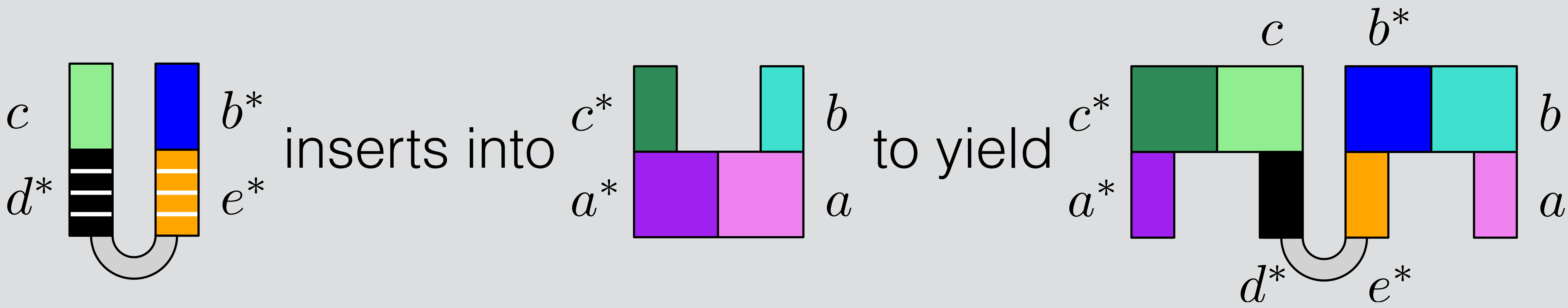
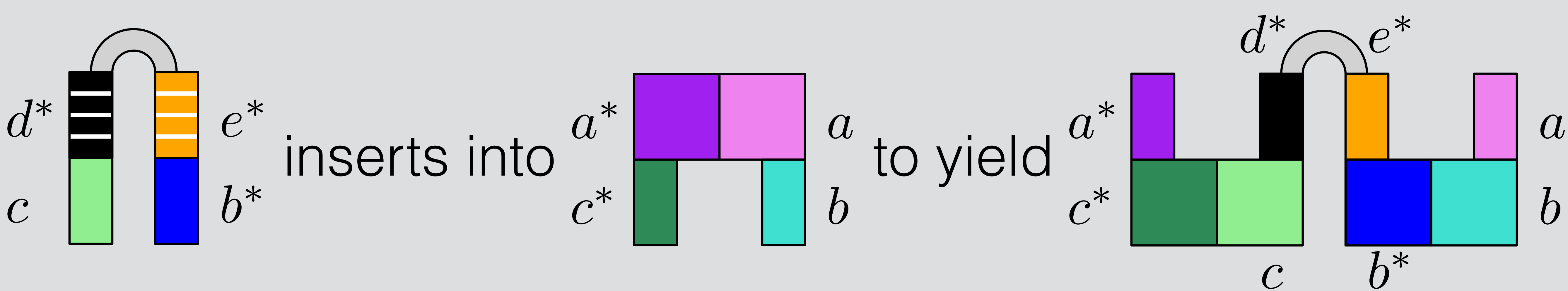


Definitions and examples

Insertions



$(c, d^*, e^*, b^*)^+$ inserts into $(a^*, c^*)(b, a)$ to yield $(a^*, c^*)(c, d^*, e^*, b^*)(b, a)$



$(d^*, c, b^*, e^*)^-$ inserts into $(c^*, a^*)(a, b)$ to yield $(c^*, a^*)(d^*, c, b^*, e^*)(a, b)$

Insertion system:

Monomer types: $(1,x,2,b)^{+}$ $(x,2^*,a,3^*)^{-}$ $(3,x,4,b)^{+}$

Initiator: $(a,1^*)(b^*,a^*)$

Insertion system:

Monomer types: $(1,x,2,b)^+$ $(x,2^*,a,3^*)^-$ $(3,x,4,b)^+$

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$$(a,1^*)(b^*,a^*)$$

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$$(a,1^*)(b^*,a^*)$$
$$\Downarrow$$
$$(a,1^*)(1,x,2,b)(b^*,a^*)$$

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$$\Downarrow$$
$$(a,1^*)(1,x,2,b)(x,2^*,a,3^*)(3,x,4,b)(b^*,a^*)$$

Terminal polymer of length 5

Insertion time

- Each monomer type has a concentration in [0,1].
- Concentrations of all types in a system must sum to ≤ 1.
- An insertion occurs after time t with:
  - t an exponential random variable with rate c.
  - c is the total concentration of insertable monomers.

Insertion system:

Monomer types: $(1,x,2,b)^{+}$ $(x,2^{*},a,3^{*})^{-}$ $(3,x,4,b)^{+}$

Initiator: $(a,1^{*})(b^{*},a^{*})$

Insertion system:

Monomer types:	$(1,x,2,b)^{+}$	$(x,2^*,a,3^*)^{-}$	$(3,x,4,b)^{+}$
Concentrations:	0.25	0.25	0.5

Initiator: $(a,1^*)(b^*,a^*)$

Insertion system:

Monomer types:	$(1,x,2,b)^+$	$(x,2^*,a,3^*)^-$	$(3,x,4,b)^+$
Concentrations:	0.25	0.25	0.5

Initiator: $(a,1^*)(b^*,a^*)$

$$(a,1^*)(b^*,a^*)$$

Insertion system:

Monomer types: $(1,x,2,b)^{+}$ $(x,2^{*},a,3^{*})^{-}$ $(3,x,4,b)^{+}$

Concentrations: 0.25 0.25 0.5

Initiator: $(a,1^{*})(b^{*},a^{*})$

$$(a,1^{*})(b^{*},a^{*})$$
$$\Downarrow t_1$$
$$(a,1^{*})(1,x,2,b)(b^{*},a^{*})$$
$$\Downarrow t_2$$
$$(a,1^{*})(1,x,2,b)(x,2^{*},a,3^{*})(b^{*},a^{*})$$
$$\Downarrow t_3$$
$$(a,1^{*})(1,x,2,b)(x,2^{*},a,3^{*})(3,x,4,b)(b^{*},a^{*})$$

Terminal polymer of length 5

Insertion system:

Monomer types:	$(1,x,2,b)^+$	$(x,2^*,a,3^*)^-$	$(3,x,4,b)^+$
Concentrations:	0.25	0.25	0.5

Initiator: $(a,1^*)(b^*,a^*)$

$$(a,1^*)(b^*,a^*)$$
$$\Downarrow t_1$$
$$(a,1^*)(1,x,2,b)(b^*,a^*)$$
$$\Downarrow t_2$$
$$(a,1^*)(1,x,2,b)(x,2^*,a,3^*)(b^*,a^*)$$
$$\Downarrow t_3$$
$$(a,1^*)(1,x,2,b)(x,2^*,a,3^*)(3,x,4,b)(b^*,a^*)$$

Terminal polymer of length 5

Expected time: $t_1 + t_2 + t_3$, with

$E[t_1] = E[t_2] = 4$, $E[t_3] = 2$.

$4 + 4 + 2 = 12$

Insertion system:

Monomer types:	$(1^*,2,2,1^*)^+$	$(x,0^*,2^*,x)^-$	$(x,2^*,0,x)^-$
Concentrations:	0.5	0.1	0.4

Initiator: $(0,1)(1,0^*)$

$(0,1)(1,0^*)$

Insertion system:

Monomer types:	$(1^*,2,2,1^*)^+$	$(x,0^*,2^*,x)^-$	$(x,2^*,0,x)^-$
Concentrations:	0.5	0.1	0.4

Initiator: $(0,1)(1,0^*)$

$$(0,1)(1,0^*)$$
$$\Downarrow t_1$$
$$(0,1)(1^*,2,2,1^*)(1,0^*)$$

Insertion system:

Monomer types: $(1^*,2,2,1^*)^+$ $(x,0^*,2^*,x)^-$ $(x,2^*,0,x)^-$

Concentrations: 0.5 0.1 0.4

Initiator: $(0,1)(1,0^*)$

$$(0,1)(1,0^*)$$
$$\Downarrow t_1$$
$$(0,1)(1^*,2,2,1^*)(1,0^*)$$
$$\Downarrow t_2$$
$$(0,1)(x,0^*,2^*,x)(1^*,2,2,1^*)(1,0^*)$$

Insertion system:

Monomer types: $(1^*,2,2,1^*)^+$ $(x,0^*,2^*,x)^-$ $(x,2^*,0,x)^-$

Concentrations: 0.5 0.1 0.4

Initiator: $(0,1)(1,0^*)$

$$
\begin{array}{c}
(0,1)(1,0^*) \\
\Downarrow t_1 \\
(0,1)(1^*,2,2,1^*)(1,0^*) \\
\Downarrow t_2 \\
(0,1)(x,0^*,2^*,x)(1^*,2,2,1^*)(1,0^*) \\
\Downarrow t_3 \\
(0,1)(x,0^*,2^*,x)(1^*,2,2,1^*)(x,2^*,0,x)(1,0^*)
\end{array}
$$

Insertion system:

Monomer types: $(1^*,2,2,1^*)^+$ $(x,0^*,2^*,x)^-$ $(x,2^*,0,x)^-$

Concentrations: 0.5 0.1 0.4

Initiator: $(0,1)(1,0^*)$

$$(0,1)(1,0^*)$$
$$\Downarrow t_1$$
$$(0,1)(1^*,2,2,1^*)(1,0^*)$$
$$\Downarrow t_2$$
$$(0,1)(x,0^*,2^*,x)(1^*,2,2,1^*)(1,0^*)$$
$$\Downarrow t_3$$
$$(0,1)(x,0^*,2^*,x)(1^*,2,2,1^*)(x,2^*,0,x)(1,0^*)$$

Terminal polymer of length 5

Insertion system:

Monomer types: $(1^*,2,2,1^*)^+$ $(x,0^*,2^*,x)^-$ $(x,2^*,0,x)^-$

Concentrations: 0.5 0.1 0.4

Initiator: $(0,1)(1,0^*)$

$$
\begin{aligned}
&(0,1)(1,0^*)\\
&\Downarrow t_1\\
&(0,1)(1^*,2,2,1^*)(1,0^*)\\
&\Downarrow t_2\\
&(0,1)(x,0^*,2^*,x)(1^*,2,2,1^*)(1,0^*)\\
&\Downarrow t_3\\
&(0,1)(x,0^*,2^*,x)(1^*,2,2,1^*)(x,2^*,0,x)(1,0^*)
\end{aligned}
$$

Terminal polymer of length 5

Expected time: $t_1 + \max(t_2, t_3)$, with

$E[t_1] = 2$, $E[t_2] = 10$, $E[t_3] = 2.5.$

$2 + 10.5 = 12.5$

Insertion system:

Monomer types:	$(1,x,2,b)^{+}$	$(x,2^*,a,1^*)^{-}$	$(x,2^*,a,x)^{-}$
Concentrations:	0.5	0.4	0.1

Initiator: $(a,1^*)(b^*,a^*)$

$$(a,1^*)(b^*,a^*)$$

Insertion system:

Monomer types: $(1,x,2,b)^+$ $(x,2^*,a,1^*)^-$ $(x,2^*,a,x)^-$

Concentrations: 0.5 0.4 0.1

Initiator: $(a,1^*)(b^*,a^*)$

$$(a,1^*)(b^*,a^*)$$
$$\Downarrow t_1$$
$$(a,1^*)(1,x,2,b)(b^*,a^*)$$

Insertion system:

Monomer types:	$(1,x,2,b)^+$	$(x,2^*,a,1^*)^-$	$(x,2^*,a,x)^-$
Concentrations:	0.5	0.4	0.1

Initiator: $(a,1^*)(b^*,a^*)$

$$(a,1^*)(b^*,a^*)$$
$$\Downarrow t_1$$
$$(a,1^*)(1,x,2,b)(b^*,a^*)$$
$$\Downarrow t_2$$
$$(a,1^*)(1,x,2,b)(x,2^*,a,x)(b^*,a^*)$$

Insertion system:

Monomer types: $(1,x,2,b)^+$ $(x,2^*,a,1^*)^-$ $(x,2^*,a,x)^-$

Concentrations: 0.5 0.4 0.1

Initiator: $(a,1^*)(b^*,a^*)$

$$(a,1^*)(b^*,a^*)$$
$$\Downarrow t_1$$
$$(a,1^*)(1,x,2,b)(b^*,a^*)$$
$$\Downarrow t_2$$
$$(a,1^*)(1,x,2,b)(x,2^*,a,x)(b^*,a^*)$$

Expected time: $t_1 + t_2$*, with*

$E[t_1] = E[t_2] = 2.$

$2 + 2 = 4$

Insertion system:

Monomer types: $(1,x,2,b)^+$ $(x,2^*,a,1^*)^-$ $(x,2^*,a,x)^-$

Concentrations: 0.5 0.4 0.1

Initiator: $(a,1^*)(b^*,a^*)$

$$
\begin{gathered}
(a,1^*)(b^*,a^*) \\
\Downarrow t_1 \\
(a,1^*)(1,x,2,b)(b^*,a^*) \\
\Downarrow t_2 \\
(a,1^*)(1,x,2,b)(x,2^*,a,x)(b^*,a^*)
\end{gathered}
$$

Insertion system:

Monomer types: $(1,x,2,b)^+$ $(x,2^*,a,1^*)^-$ $(x,2^*,a,x)^-$

Concentrations: 0.5 0.4 0.1

Initiator: $(a,1^*)(b^*,a^*)$

$$(a,1^*)(b^*,a^*)$$
$$\Downarrow t_1$$
$$(a,1^*)(1,x,2,b)(b^*,a^*)$$
$$\Downarrow t_2$$
$$(a,1^*)(1,x,2,b)(x,2^*,a,1^*)(b^*,a^*)$$

Insertion system:

Monomer types: $(1,x,2,b)^+$ $(x,2^*,a,1^*)^-$ $(x,2^*,a,x)^-$

Concentrations: 0.5 0.4 0.1

Initiator: $(a,1^*)(b^*,a^*)$

$$(a,1^*)(b^*,a^*)$$
$$\Downarrow t_1$$
$$(a,1^*)(1,x,2,b)(b^*,a^*)$$
$$\Downarrow t_2$$
$$(a,1^*)(1,x,2,b)(x,2^*,a,1^*)(b^*,a^*)$$
$$\Downarrow t_3$$
$$(a,1^*)(1,x,2,b)(x,2^*,a,1^*)(1,x,2,b)(b^*,a^*)$$

Insertion system:

Monomer types: $(1,x,2,b)^{+}$ $(x,2^*,a,1^*)^{-}$ $(x,2^*,a,x)^{-}$

Concentrations: 0.5 0.4 0.1

Initiator: $(a,1^*)(b^*,a^*)$

$$
\begin{gathered}
(a,1^*)(b^*,a^*)\\
\Downarrow t_1\\
(a,1^*)(1,x,2,b)(b^*,a^*)\\
\Downarrow t_2\\
(a,1^*)(1,x,2,b)(x,2^*,a,1^*)(b^*,a^*)\\
\Downarrow t_3\\
(a,1^*)(1,x,2,b)(x,2^*,a,1^*)(1,x,2,b)(b^*,a^*)\\
\Downarrow t_4\\
(a,1^*)(1,x,2,b)(x,2^*,a,1^*)(1,x,2,b)(x,2^*,a,x)(b^*,a^*)
\end{gathered}
$$

Insertion system:

Monomer types: $(1,x,2,b)^+$ $(x,2^*,a,1^*)^-$ $(x,2^*,a,x)^-$

Concentrations: 0.5 0.4 0.1

Initiator: $(a,1^*)(b^*,a^*)$

$$(a,1^*)(b^*,a^*)$$
$$\Downarrow t_1$$
$$(a,1^*)(1,x,2,b)(b^*,a^*)$$
$$\Downarrow t_2$$
$$(a,1^*)(1,x,2,b)(x,2^*,a,1^*)(b^*,a^*)$$
$$\Downarrow t_3$$
$$(a,1^*)(1,x,2,b)(x,2^*,a,1^*)(1,x,2,b)(b^*,a^*)$$
$$\Downarrow t_4$$
$$(a,1^*)(1,x,2,b)(x,2^*,a,1^*)(1,x,2,b)(x,2^*,a,x)(b^*,a^*)$$

Expected time: $t_1 + t_2 + t_3 + t_4$*, with*

$E[t_1] = E[t_2] = E[t_3] = E[t_4] = 2.$

$2 + 2 + 2 + 2 = 8$

Expressive power

Expressive Power

- Insertion systems: initiator + set of monomers = set of polymers, with terminal polymer subset.
- Context-free grammars: start symbol + set of rules = set of partial derivations, with string subset.
- Do insertion systems and context-free grammars have equal “expressive-ness”?

Expressive Power

- Insertion systems: initiator + set of monomers = set of polymers, with terminal polymer subset.

- Context-free grammars: start symbol + set of rules = set of partial derivations, with string subset.

- Do insertion systems and context-free grammars have equal “expressive-ness”? *Yes.*

Expressive Power

Theorem: every insertion system can be expressed as a context-free grammar. [Dabby, Chen 2013]

Theorem: every context-free grammar can be expressed as an insertion system. [This work]

Expressive Power

Theorem: every insertion system can be expressed as a context-free grammar. [Dabby, Chen 2013]

Theorem: every context-free grammar can be expressed as an insertion system. [This work]

Insertion system:

Monomer types: $(1,x,2,b)^+$ $(x,2^*,a,3)^-$ $(3,x,4,b)^+$

Concentrations: 0.25 0.25 0.25

Initiator: $(a,1)(b^*,a^*)$

Context-free grammar:

Production rules:

$S_{(a,1)(b^*,a^*)} \to S_{(a,1)(1,x)} S_{(2,b)(b^*,a^*)}$

$S_{(2,b)(b^*,a^*)} \to S_{(2,b)(x,2^*)} S_{(a,3)(b^*,a^*)}$

$S_{(a,3)(b^*,a^*)} \to S_{(a,3)(3,x)} S_{(4,b)(b^*,a^*)}$

$S_{(a,1)(1,x)} \to (a, 1)(1, x)$

$S_{(2,b)(x,2^*)} \to (2,b)(x,2^*)$

$S_{(a,3)(3,x)} \to (a,3)(3,x)$

$S_{(4,b)(x,4^*)} \to (4,b)(x,4^*)$

Start symbol: $S_{(a,1)(b^*,a^*)}$

Expressive Power

<u>Theorem:</u> every insertion system can be expressed as a context-free grammar. [Dabby, Chen 2013]

<u>Theorem:</u> every context-free grammar can be expressed as an insertion system. [This work]

Expressive Power

Theorem: every insertion system can be expressed as a context-free grammar. [Dabby, Chen 2013]

Theorem: every context-free grammar can be expressed as an insertion system. [This work]

Turning rules into monomers

Rule: $A \rightarrow B\ C$

Derivation step: $cdADee \Rightarrow cdBCDee$

Monomer type: $(1,x,2,b)^{+}$

Insertion: $(a,1)(b^{*},a^{*}) \Rightarrow (a,1)(1,x,2,b)(b^{*},a^{*})$

Rules completely replace non-terminals.
Insertions do not completely replace insertion sites.

Turning rules into monomers

Rule: $A_2 \rightarrow A_4 A_5$

Derivation step: $A_1 A_2 A_3 \Rightarrow A_1 A_4 A_5 A_3$

(n non-terminals total)

Insertion site: $u^*, s_a^*)(s_d, u$ with $a+d \bmod n = 2$

Site modification: $u^*, s_a^*)(s_d, u$

$$\Downarrow$$

$$\vdots$$

$$\Downarrow$$

$u^*, s_a^*)(s_b, u, \ldots, u^*, s_c^*)(s_d, u$

with $a+b \bmod n = 4$, $c+d \bmod n = 5$

Monomer types: $\Theta(n)$ per rule.

Polymer lengths

Insertion system:

Monomer types: $(1,x,2,b)^{+}$ $(x,2^{*},a,1^{*})^{-}$ $(x,2^{*},a,x)^{-}$

Concentrations: 0.5 0.4 0.1

Initiator: $(a,1^{*})(b^{*},a^{*})$

$(a,1^{*})(1,x,2,b)(x,2^{*},a,x)(b^{*},a^{*})$

$(a,1^{*})(1,x,2,b)(x,2^{*},a,1^{*})(1,x,2,b)(x,2^{*},a,x)(b^{*},a^{*})$

$(a,1^{*})(1,x,2,b)(x,2^{*},a,1^{*})(1,x,2,b)(x,2^{*},a,1^{*})(1,x,2,b)(x,2^{*},a,x)(b^{*},a^{*})$

•••

Context-free grammar:

Production rules: $S \rightarrow A$ $A \rightarrow aaA$ $A \rightarrow aa$

Start symbol: S

aa

aaaa

aaaaaa

•••

Insertion system:

Monomer types: $(1,x,2,b)^+$ $(x,2^*,a,1^*)^-$ $(x,2^*,a,x)^-$

Concentrations: 0.5 0.4 0.1

Initiator: $(a,1^*)(b^*,a^*)$

$(a,1^*)(1,x,2,b)(x,2^*,a,x)(b^*,a^*)$

$(a,1^*)(1,x,2,b)(x,2^*,a,1^*)(1,x,2,b)(x,2^*,a,x)(b^*,a^*)$

$(a,1^*)(1,x,2,b)(x,2^*,a,1^*)(1,x,2,b)(x,2^*,a,1^*)(1,x,2,b)(x,2^*,a,x)(b^*,a^*)$

•••

Constructing arbitrarily long polymers is easy if infinite polymers allowed.

Context-free grammar:

Production rules: $S \rightarrow A$ $A \rightarrow aaA$ $A \rightarrow aa$

Start symbol: S

aa

aaaa

aaaaaa

•••

Constructing long polymers

Theorem: a system with k monomer types constructing a finite number of polymers can construct:

- polymers of length $2^{\Theta(k^{1/2})}$ [Dabby, Chen 2013]
- polymers of length $2^{\Theta(k^{3/2})}$ [This work]
- only polymers of length $2^{O(k^{3/2})}$ [This work]

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- only polymers of length $2^{O(k^{3/2})}$ [This work]

Constructing long polymers

- Consider *insertion sequences*: repeated insertions into the site resulting from previous insertion.

$$\ldots u,0)(0,u^*\ldots$$

$$\Downarrow$$

$$\ldots u,0)(0^*,1,x,0^*)(0,u^*\ldots$$

$$\Downarrow$$

$$\ldots u,0)(x,u^*,1^*,1^*)(0^*,1,x,0^*)(0,u^*\ldots$$

$$\Downarrow$$

$$\ldots u,0)(x,u^*,1^*,1^*)(1,\ x,\ 2,\ 0)(0^*,1,x,0^*)(0,u^*\ldots$$

Constructing long polymers

- Consider *insertion sequences*: repeated insertions into the site resulting from previous insertion.

…u,0)(0,u*…

⇓

…u,0)(0*,1…

⇓

…1*,1*)(0*,1…

⇓

…2, 0)(0*,1…

Constructing long polymers

- Consider *insertion sequences*: repeated insertions into the site resulting from previous insertion.

$$
\begin{gathered}
\ldots u,0)(0,u^{*}\ldots \\
\Downarrow \\
\ldots u,0)(0^{*},1\ldots \\
\Downarrow \\
\ldots 1^{*},1^{*})(0^{*},1\ldots \\
\Downarrow \\
\ldots 2,\ 0)(0^{*},1\ldots
\end{gathered}
$$

Constructing long polymers

- Consider *insertion sequences*: repeated insertions into the site resulting from previous insertion.
- Ingredient 1: long insertion sequence with no repeated insertion sites.

…0,0)(0,0*…

$\Downarrow$ O(1)

…0,0)(1,0*…

$\Downarrow$ O(1)

⋮

$\Downarrow$ O(1)

…r, r)(r,r*…

$\Theta(k^{3/2})$

Constructing long polymers

- Consider *insertion sequences*: repeated insertions into the site resulting from previous insertion.

- Ingredient 1: long insertion sequence with no repeated insertion sites.

- Ingredient 2: duplication of each site in sequence.

$$\ldots 0,0)(0,0^*\ldots$$
$$\Downarrow O(1)$$
$$\ldots 0,0)(1,0^*\ldots 0,0)(1,0^*\ldots$$
$$\Downarrow O(1)$$
$$\ldots 0,\ 0)(2,0^*\ldots 0,0)(2,0^*\ldots 0,0)(2,0^*\ldots 0,0)(2,0^*\ldots$$

Constructing long polymers

- Consider *insertion sequences*: repeated insertions into the site resulting from previous insertion.
- Ingredient 1: long insertion sequence
  with no repeated insertion sites.
- Ingredient 2: duplication of each site in sequence.
- Combine these for long polymers.

Constructing long polymers

…0,0)(0,0*…

$\Downarrow$ O(1)

…0,0)(1,0*…0,0)(1,0*…

$\Downarrow$ O(1)

…0, 0)(2,0*…0,0)(2,0*…0,0)(2,0*…0,0)(2,0*…

$\Downarrow$ O(1)

⋮

$\Downarrow$ O(1)

…r, r)(r,r*…r, r)(r,r*…r, r)(r,r*…r,r)(r,r*…r,r)(r,r*…r,r)(r,r*…

$\Theta(k^{3/2})$

$2^{\Theta(k^{3/2})}$

Constructing long polymers

- Ingredient 1: long insertion sequence with no repeated insertion sites.

$$s_a, s_b)(s_c, s_a{}^*$$

variables

Use $r+1 = \Theta(k^{1/2})$ values of a, b, c $\Rightarrow \Theta(r^3) = \Theta(k^{3/2})$ insertion sites.

Case:	$b < r$	$b = r,\ c < r$	$b = c = r,\ a < r$
Insertion sequence:	$s_a, s_b)(s_c, s_a{}^* \overset{O(1)}{\Downarrow} s_a, s_{b+1})(s_c, s_a{}^*$	$s_a, s_r)(s_c, s_a{}^* \overset{O(1)}{\Downarrow} s_a, s_0)(s_{c+1}, s_a{}^*$	$s_a, s_r)(s_r, s_a{}^* \overset{O(1)}{\Downarrow} s_{a+1}, s_0)(s_0, s_{a+1}{}^*$
Result:	$++b$	$b = 0,\ ++c$	$b = c = 0,\ ++a$

Constructing long polymers

- Ingredient 1: long insertion sequence with no repeated insertion sites.

$$s_a, s_b)(s_c, s_a{}^*$$

variables

Use $r+1 = \Theta(k^{1/2})$ values of a, b, c $\Rightarrow \Theta(r^3) = \Theta(k^{3/2})$ insertion sites.

Case:	$b < r$	$b = r,\ c < r$	$b = c = r,\ a < r$
Insertion sequence:	$s_a, s_b)(s_c, s_a{}^* \Downarrow_{O(1)} s_a, s_{b+1})(s_c, s_a{}^*$	$s_a, s_r)(s_c, s_a{}^* \Downarrow_{O(1)} s_a, s_0)(s_{c+1}, s_a{}^*$	$s_a, s_r)(s_r, s_a{}^* \Downarrow_{O(1)} s_{a+1}, s_0)(s_0, s_{a+1}{}^*$
Result:	++b	b = 0, ++c	b = c = 0, ++a

Constructing long polymers

$\ldots s_a, s_r)(s_c, s_a{}^* \ldots$

Case:	$b = r,\ c < r$
Insertion sequence:	$s_a, s_r)(s_c, s_a{}^* \Downarrow_{O(1)} s_a, s_0)(s_{c+1}, s_a{}^*$
Result:	$b = 0,\ {+}{+}c$

$\ldots s_a, s_0{}^*)(s_{c+1}{}^*,\ s_a{}^* \ldots$

Constructing long polymers

Case:	$b = r,\ c < r$
Insertion sequence:	$s_a, s_r)(s_c, s_a^* \overset{O(1)}{\Downarrow} s_a, s_0)(s_{c+1}, s_a^*$
Result:	$b = 0,\ {+}{+}c$

$$
\begin{array}{c}
\ldots s_a, s_r)(s_c, s_a^* \ldots \\
\Downarrow \\
\ldots s_a, s_r)(s_r^*, s_{f(c)} \ldots \\
\Downarrow \\
\ldots s_a, s_r)(s_{f'(c)},\ s_a^* \ldots \\
\Downarrow \\
\ldots s_{f''(c)}, s_{f'(c)}^*)(s_{f'(c)},\ s_a^* \ldots \\
\Downarrow \\
\ldots s_a, s_0^*)(s_{f'(c)},\ s_a^* \ldots \\
\Downarrow \\
\ldots s_a, s_0^*)(s_{f'(c)},\ s_a^* \ldots \\
\Downarrow \\
\ldots s_a, s_0^*)(s_{f'(c)},\ s_a^* \ldots \\
\Downarrow \\
\ldots s_a, s_0^*)(s_0^*,\ s_{f'''(c+1)}^* \ldots \\
\Downarrow \\
\ldots s_a, s_0^*)(s_{c+1}^*,\ s_a^* \ldots
\end{array}
$$

Constructing long polymers

Case:	$b = r,\ c < r$
Insertion sequence:	$s_a,s_r)(s_c,s_a^* \Downarrow_{O(1)} s_a,s_0)(s_{c+1},s_a^*$
Result:	$b = 0,\ {+}{+}c$

This is the easiest case.

A total of 23 monomer type families used.

$$\ldots s_a,s_r)(s_c,s_a^*\ldots$$
$$\Downarrow$$
$$\ldots s_a,s_r)(s_r^*,s_{f(c)}\ldots$$
$$\Downarrow$$
$$\ldots s_a,s_r)(s_{f'(c)},\ s_a^*\ldots$$
$$\Downarrow$$
$$\ldots s_{f''(c)},s_{f'(c)}^*)(s_{f'(c)},\ s_a^*\ldots$$
$$\Downarrow$$
$$\ldots s_a,s_0^*)(s_{f'(c)},\ s_a^*\ldots$$
$$\Downarrow$$
$$\ldots s_a,s_0^*)(s_{f'(c)},\ s_a^*\ldots$$
$$\Downarrow$$
$$\ldots s_a,s_0^*)(s_{f'(c)},\ s_a^*\ldots$$
$$\Downarrow$$
$$\ldots s_a,s_0^*)(s_0^*,\ s_{f'''(c+1)}^*\ldots$$
$$\Downarrow$$
$$\ldots s_a,s_0^*)(s_{c+1}^*,\ s_a^*\ldots$$

An upper bound for polymer length

- Every insertion sequence in polymer must consist of unique insertion sites **a,b)(c,a*** that each accept a monomer.
- Prove there are $O(k^{3/2})$ such sites.
- Proof idea: maximize |{a}|*|{b}|*|{c}| subject to
    - |{a}|*|{b}|, |{a}|*|{c}| = O(k) since they are monomer right/left halves.
    - |{b}|*|{c}| = O(k) since site needs $(b^*,\_,\_,c^*)^+$ inserted.
- |{a}|*|{b}|*|{c}| = $O(k^{3/2})$

Polymer growth speed

Insertion system:

Monomer types: $(b^*,a^*,a,b)^+$ $(b^*,x,x,b)^+$

Concentrations: 0.5 0.5

Initiator: $(a,b)(b^*,a^*)$

$$(a,b)(b^*,a^*)$$
$$\Downarrow 1$$
$$(a,b)(b^*,a^*,a,b)(b^*,a^*)$$
$$\Downarrow 2$$
$$(a,b)(b^*,a^*,a,b)(b^*,a^*,a,b)(b^*,a^*,a,b)(b^*,a^*)$$
$$\Downarrow 4$$
$$(a,b)(b^*,x,x,b)(b^*,a^*,a,b)(b^*,x,x,b)(b^*,a^*,a,b)(b^*,x,x,b)(b^*,a^*,a,b)(b^*,x,x,b)(b^*,a^*)$$

Each round of insertions takes O(1) expected time.

Construction of length $n = 2^i-1$ takes O(i) expected time.

O(log(n))

Insertion system:

Monomer types: $(b^*,a^*,a,b)^+$ $(b^*,x,x,b)^+$

Concentrations: 0.5 0.5

Initiator: $(a,b)(b^*,a^*)$

$$(a,b)(b^*,a^*)$$
$$\Downarrow 1$$
$$(a,b)(b^*,a^*,a,b)(b^*,a^*)$$
$$\Downarrow 2$$
$$(a,b)(b^*,a^*,a,b)(b^*,a^*,a,b)(b^*,a^*,a,b)(b^*,a^*)$$
$$\Downarrow$$
$$(a,b)(b^*,x,x,b)(b^*,a^*,a,b)(b^*,x,x,b)(b^*,a^*,a,b)(b^*,x,x,b)(b^*,a^*,a,b)(b^*,x,x,b)(b^*,a^*)$$

Constructing polymers in $O(\log(n))$ expected time is easy if infinite polymers allowed.

Each round of insertions takes $O(1)$ expected time.

Construction of length $n = 2^i-1$ takes $O(i)$ expected time.

$O(\log(n))$

Constructing polymers quickly

Theorem: a system constructing a finite number of polymers can **deterministically** construct a polymer of length n in:

- $O(\log^3(n))$ expected time [Dabby, Chen 2013]
- $O(\log^{5/3}(n))$ expected time [This work]
- only $\Omega(\log^{5/3}(n))$ expected time [This work]

Constructing polymers quickly

Theorem: a system constructing a finite number of polymers can **deterministically** construct a polymer of length n in:

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- $O(\log^{5/3}(n))$ expected time [This work]
- only $\Omega(\log^{5/3}(n))$ expected time [This work]

A lower bound for growth speed

- Consider longest insertion sequence carried out;
  must have length at least $\log_2(n)$.

- By polymer length upper bound, must insert
  $\Omega(\log^{2/3}(n))$ different monomer types.

- Optimization problem: pick #insertions, concentrations of $\log^{2/3}(n)$ monomer types into $\log_2(n)$ sites to minimize total expected time.

- Assuming each site accepts 1 monomer type,
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System is *deterministic.*

Constructing polymers quickly

Theorem: a system constructing a finite number of polymers can **deterministically** construct a polymer of length n in:

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Speedup by non-determinism

$$\ldots s_1, s_0)(s_0^*, s_1^* \ldots$$

$$\Downarrow_{O(1)}$$

$$\ldots s_1, s_1)(s_1^*, s_1^* \ldots$$

Needs unique monomers.

Unique monomers ⇒ slow.

Speedup by non-determinism

$$\ldots s_6, s_3)(s_3^*, s_6^* \ldots$$

$$\Downarrow_{O(1)}$$

$$\ldots s_6, s_4)(s_4^*, s_6^* \ldots$$

Needs unique monomers.

Unique monomers ⇒ slow.

Speedup by non-determinism

$\ldots s_6, s_3)(s_3{}^*, s_6{}^* \ldots$

Speedup by non-determinism

$$\ldots s_6, s_3)(s_3^*, s_6^* \ldots$$

$$\Downarrow \text{ Guess b}$$

$$\ldots s_6, s_3)(s_{f(b+1)}^*, s_6^* \ldots$$

Speedup by non-determinism

$$\ldots s_6, s_3)(s_3^*, s_6^* \ldots$$

$$\Downarrow \text{ Guess b}$$

$$\ldots s_6, s_3)(s_{f(b+1)}^*, s_6^* \ldots$$

$$\Downarrow \text{ If b = 3, guess a}$$

$$\ldots s_{f(a)}, s_{f(b+1)})(s_{f(b+1)}^*, s_6^* \ldots$$

Speedup by non-determinism

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$$\Downarrow \text{ If b = 3, guess a}$$

$$\ldots s_{f(a)}, s_{f(b+1)})(s_{f(b+1)}{}^*, s_6{}^* \ldots$$

$$\Downarrow \text{ If a = 6}$$

$$\ldots s_{f(a)}, s_{f(b+1)})(s_{f(b+1)}{}^*, s_{f(a)}{}^* \ldots$$

Speedup by non-determinism

$$\ldots s_6, s_3)(s_3{}^*, s_6{}^* \ldots$$

$$\Big\Downarrow \text{ Guess b}$$

$$\ldots s_6, s_3)(s_{f(b+1)}{}^*, s_6{}^* \ldots$$

$$\Big\Downarrow \text{ If b = 3, guess a}$$

$$\ldots s_{f(a)}, s_{f(b+1)})(s_{f(b+1)}{}^*, s_6{}^* \ldots$$

$$\Big\Downarrow \text{ If a = 6}$$

$$\ldots s_{f(6)}, s_{f(4)})(s_{f(4)}{}^*, s_{f(6)}{}^* \ldots$$

Speedup by non-determinism

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Speedup by non-determinism

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Speedup by non-determinism

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$$\Downarrow \text{ Guess b}$$

$$\ldots s_6, s_3)(s_{f(b+1)}{}^*, s_6{}^* \ldots$$

$$\Downarrow \text{ If } b \neq 3\text{, insertion sequence done.}$$

$$\times$$

Speedup by non-determinism

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Speedup by non-determinism

$$\ldots s_6, s_3)(s_3^*, s_6^* \ldots$$

$$\Downarrow \text{ Guess b}$$

$$\ldots s_6, s_3)(s_{f(b+1)}^*, s_6^* \ldots$$

$$\Downarrow \text{ If } b = 3, \text{ guess a}$$

$$\ldots s_{f(a)}, s_{f(b+1)})(s_{f(b+1)}^*, s_6^* \ldots$$

$$\Downarrow \text{ If } a \neq 6, \text{ insertion sequence done.}$$

$$\times$$

Why does guessing give a speed-up?

Speedup by non-determinism

$$\ldots S_6, S_3)(S_3{}^*, S_6{}^* \ldots$$

$$\Downarrow \text{Guess b}$$

$$\ldots S_6, S_3)(S_{f(b+1)}{}^*, S_6{}^* \ldots$$

$$\Downarrow \text{If b = 3, guess a}$$

$$\ldots S_{f(a)}, S_{f(b+1)})(S_{f(b+1)}{}^*, S_6{}^* \ldots$$

$$\Downarrow \text{If a = 6}$$

$$\ldots S_{f(a)}, S_{f(b+1)})(S_{f(b+1)}{}^*, S_{f(a)}{}^* \ldots$$

Speedup by non-determinism

$s_6, s_3)(s_3{}^*, s_6{}^*$

$\Downarrow$ Guess b

$s_6, s_3)(s_{f(b+1)}{}^*, s_6{}^*$

$s_6, s_5)(s_5{}^*, s_6{}^*$

$\Downarrow$ Guess b

$s_6, s_5)(s_{f(b+1)}{}^*, s_6{}^*$

$s_6, s_8)(s_8{}^*, s_6{}^*$

$\Downarrow$ Guess b

$s_6, s_8)(s_{f(b+1)}{}^*, s_6{}^*$

$s_6, s_{11})(s_{11}{}^*, s_6{}^*$

$\Downarrow$ Guess b

$s_6, s_{11})(s_{f(b+1)}{}^*, s_6{}^*$

Speedup by non-determinism

$s_6,s_3)(s_3{}^*,s_6{}^*$ $\quad$ $s_6,s_5)(s_5{}^*,s_6{}^*$

$\Downarrow$ Guess b $\quad$ $\Downarrow$ Guess b

$s_6,s_3)(s_{f(b+1)}{}^*,s_6{}^*$ $\quad$ $s_6,s_5)(s_{f(b+1)}{}^*,s_6{}^*$

The same set of monomers are insertable to *all* of these sites.

$s_6,s_8)(s_8{}^*,s_6{}^*$ $\quad$ $s_6,s_{11})(s_{11}{}^*,s_6{}^*$

$\Downarrow$ Guess b $\quad$ $\Downarrow$ Guess b

$s_6,s_8)(s_{f(b+1)}{}^*,s_6{}^*$ $\quad$ $s_6,s_{11})(s_{f(b+1)}{}^*,s_6{}^*$

Non-determinism speed-up

- Sites $s_a, s_b)(s_b^*, s_a^*$ with $\Theta(k^{1/2})$ values for a, b
    - So $k = \Theta(\log(n))$
- Every site accepts $\Theta(k^{1/2})$ monomers: $\Omega(1/k^{1/2})$ total concentration.
- Expected insertion time is $O(k^{1/2}) = O(\log^{1/2}(n))$.
- Total expected construction time is $O(\log^{3/2}(n))$.

Polymer length vs. growth speed

Types vs Speed Tradeoff

Is it possible to construct a polymer using $O(\log^{2/3}(n))$ monomer types in $O(\log^{3/2}(n))$ expected time?

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Theorem: in a system constructing a finite number of polymers, constructing a polymer of length n using k monomer types takes $\Omega(\log^2(n)/k^{1/2})$ expected time.

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<u>Theorem:</u> in a system constructing a finite number of polymers, constructing a polymer of length n using k monomer types takes $\Omega(\log^2(n)/k^{1/2})$ expected time.

> $O(\log^{2/3}(n))$ types $\Rightarrow$ $\Omega(\log^{5/3}(n))$ expected time
>
> $O(\log^{3/2}(n))$ expected time $\Rightarrow$ $\Omega(\log(n))$ types

Conclusions

Conclusions

- Insertion systems and context-free grammars are computationally equivalent.
- Growing finitely takes more monomer types and more time than infinite growth.
- Growing the longest finite polymers possible requires use of novel properties — cannot use CFG equivalence!
- Growing finite polymers fastest needs non-determinism and more monomer types.
- A smooth tradeoff between polymer length and growth rate.

Thank you.

Joint work with Benjamin Hescott and Caleb Malchik.

Some results published in:

C. Malchik, A. Winslow, Tight bounds for active self-assembly with an insertion primitive, Proceedings of 22nd European Symposium on Algorithms, 677-688, 2014.