Polycube Unfoldings Satisfying Conway's Criterion

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Abstract

We prove that some classes of polycubes admit vertex and edge unfoldings that tile the plane. Surprisingly, every such polycube found has an unfolding satisfying *Conway's criterion* and it remains open whether all polycubes have edge unfoldings satisfying Conway's criterion.

1 Introduction

It is mathematical folklore that the cube can be unfolded to flat by cutting along its edges in 11 non-congruent ways, and that all 11 such unfoldings are non-overlapping *edge unfoldings* tile the plane. Akiyama [1] calls such a polyhedron a *semi-tile-maker* and conjectured that the cube and octahedron are the only two (weakly) convex semi-tile-makers. Shephard [8] introduced the more general class of *tesselation polyhedra* that admit at least one edge unfolding that tiles the plane. They characterize the convex tesselation polyhedra with regular-polygon faces.

Here we mainly consider the existence of tesselation *polycubes*: shapes formed by a union of face-adjacent cubes. Clearly, an edge unfolding is a prerequisite for a tesselation polyhedron. But as with convex polyhedra, it remains open whether every polycube even admits an edge unfolding (Open Problems 21.11 and 22.15 of [5]). The same is true even if the unfolded shape need only be connected by single vertices, so-called *vertex unfoldings*. Prior work on unfolding "gridded" orthogonal polyhedra implies that at least all genus-0 polycubes have vertex unfoldings [4] and a general class of monotone "orthostack" polycubes have edge unfoldings [3].

We obtain three positive results on the existence of polycubes with vertex or edge unfoldings that tile the plane. Curiously, every polycube we find with such an unfolding also has one satisfying Conway's criterion [7], a sufficient condition for tiling the plane that produces tilings with 180° rotational symmetry. Even more curious, the same is true of the tesslation polyhedron found by Akiyama et al. [2]!

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2 Definitions

Polyominoes and polycubes. A *polyomino* is an orthogonal polygon whose edges have identical length; alternatively, a polygon consisting of the edge-connected union of congruent squares. A *polycube* is the three-dimensional equivalent of a polyomino.

Unfoldings. An *edge unfolding* of a polycube is a development of the polycube's surface to a plane as a polyomino whose boundary consists of edges of the polycube. A *vertex unfolding* of a polycube is an edge unfolding where the development must be connected, but not necessarily a polyomino.

Conway's criterion. A polyomino P has a *plane tiling* provided that the plane is the union of (infinitely many) interior-disjoint congruent copies of P. The *boundary word* of a simply connected polyomino P, denoted $\mathcal{B}(P)$, is the circular word over a four-letter alphabet corresponding to directions of the edges as they appear along $\mathcal{B}(P)$. A polyomino P *satisfies Conway's criterion* (and has a plane tiling, see Figure 1) provided $\mathcal{B}(P) = ABC\widehat{A}DE$, where B, C, D, and E are palindromes and $\widehat{A}$ are the opposite directions of A in the reverse order.

3 Results

Theorem 3.1. *Every genus-0 or genus-1 polycube has a vertex unfolding satisfying Conway's criterion.*

A *k-cube* is a polycube consisting of k cubes. The next result is obtained via computer search. Similar search has yielded an unfolding of the 8-cube Dali cross (see Figure 2), answering a question posted by Diaz and O'Rourke [6].

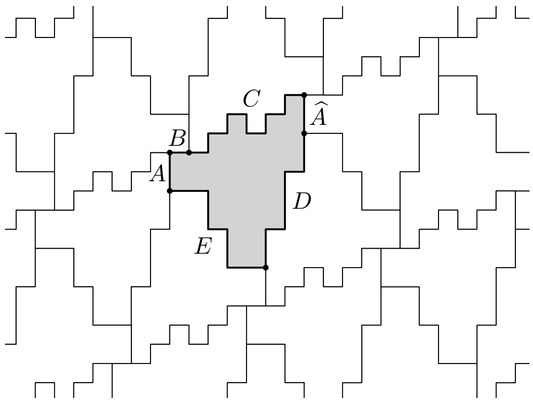

Figure 1: A polyomino satisfying Conway's criterion in the induced plane tiling.

Theorem 3.2. *Every k-cube for $k \le 7$ has an edge unfolding satisfying Conway's criterion.*

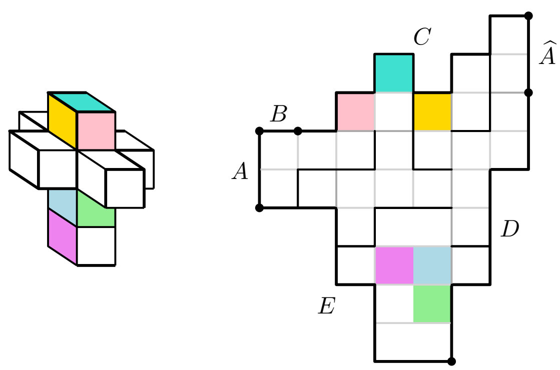

Figure 2: An unfolding of the Dali cross satisfying Conway's criterion. Light and dark gray crease indicate mountain and valley folds, respectively.

The *dual graph* of a polycube has a vertex for each cube and edges between face-adjacent cubes. A *path-like* polycube has a path dual graph. Each path in the dual graph corresponds to a path in the cubic lattice. A *one-layer* polycube has all such paths in a common plane and all length-l paths of an l-*separated* polycube have at most one turn.

Theorem 3.3. *Every path-like one-layer or path-like 4-separated polycube has an edge unfolding satisfying Conway's criterion.*

We can obtain similar results for some classes of tree-like polycubes as well, but do not know how far these results can be extended:

Problem 3.4. *Does every path-like polycube have an edge unfolding satisfying Conway's criterion?*

We lack any example of a polycube without such an unfolding, leading to a strengthened version of Open Problem 22.15 of [5]:

Problem 3.5. *Does every polycube have an edge unfolding satisfying Conway's criterion?*

More specifically, perhaps the *level-1 Menger sponge* is a counterexample: the 20-cube with cube centers at all locations (x, y, z) such that $x, y, z \in \{-1, 0, 1\}$ and at most one of x, y, z is equal to 0.

Problem 3.6. *Does the level-1 Menger sponge polycube have an edge unfolding satisfying Conway's criterion?*

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