Some Open Problems in Polyomino Tilings

Andrew Winslow[1]

University of Texas Rio Grande Valley, Edinburg, TX, USA
andrew.winslow@utrgv.edu

Abstract. The author surveys 15 open problems regarding the algorithmic, structural, and existential properties of polyomino tilings.

1 Introduction

In this work, we consider a variety of open problems related to polyomino tilings. For further reference on polyominoes and tilings, the original book on the topic by Golomb [15] (on polyominoes) and more recent book of Grünbaum and Shephard [23] (on tilings) are essential. Also see [2,26] and [19,21] for open problems in polyominoes and tiling more broadly, respectively. We begin by introducing the definitions used throughout; other definitions are introduced as necessary in later sections.

Definitions. A *polyomino* is a subset of $\mathbb{R}^2$ formed by a strongly connected union of axis-aligned unit squares centered at locations on the square lattice $\mathbb{Z}^2$.

Let $\mathscr{T} = \{T_1, T_2, \dots\}$ be an infinite set of finite simply connected closed sets of $\mathbb{R}^2$. Provided the elements of $\mathscr{T}$ have pairwise disjoint interiors and cover the Euclidean plane, then $\mathscr{T}$ is a *tiling* and the elements of $\mathscr{T}$ are called *tiles*.

Provided every $T_i \in \mathscr{T}$ is congruent to a common shape T, then $\mathscr{T}$ is *monohedral*, T is the *prototile* of $\mathscr{T}$, and the elements of $\mathscr{T}$ are called *copies* of T. In this case, T is said to *have* a tiling.

2 Isohedral Tilings

We begin with monohedral tilings of the plane where the prototile is a polyomino. If a tiling $\mathscr{T}$ has the property that for all $T_i, T_j \in \mathscr{T}$, there exists a symmetry of $\mathscr{T}$ mapping T_i to T_j, then $\mathscr{T}$ is *isohedral*; otherwise the tiling is *non-isohedral* (see Figure 1).

The enforced symmetry of isohedral tilings implies that isohedral tilings of polyominoes can be characterized by seven *boundary criteria*: either the boundary of the polyomino satisfies one of these criteria, or does not have an isohedral tiling (see [27] for further discussion). The fastest known algorithm for testing these criteria runs in $O(n \log^2 n)$ time [27], where n is the number of unit-length edges along the boundary of the polyomino. Is a faster algorithm possible?

Open Problem 1 (Open Problem 2 of [27]) *Is there an $O(n)$-time algorithm for determining whether a polyomino has an isohedral tiling?*

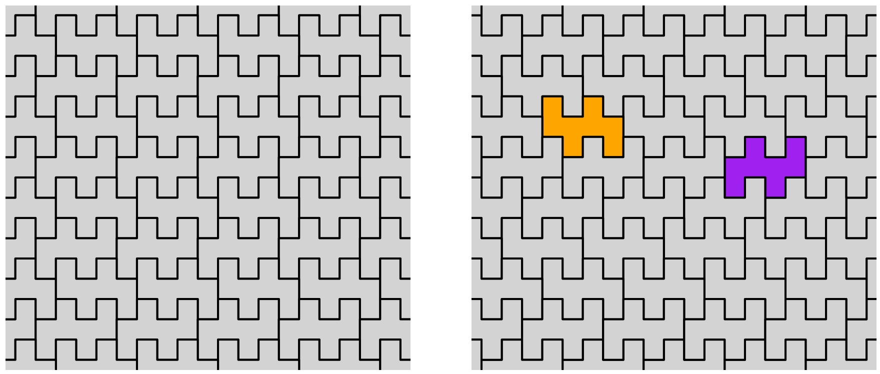

Fig. 1. Isohedral (left) and non-isohedral (right) tilings of a polyomino. There is no symmetry of the right tiling mapping one colored tile to the other.

The following problem follows from the surprising result of Wijshoff and van Leeuwen [37] and Beaquier and Nivat [3] that every polyomino (and polygon) with a tiling consisting only translated copies of the prototile also has such a tiling that is isohedral. An example of Rhoads [33] seen in Figure 3 proves that this is not the case for tilings using 90°-, 180°-, and 270°-rotated copies. However, it is unknown if this is true for tilings using other subsets of the eight possible orientations. Specifically, isohedral tilings using only translated and 180°-rotated copies of the prototile are characterized by those polyominoes satisfying *Conway's criterion* [14,34] (see 2 for an example).

Open Problem 2 (Open Problem 3 of [27]) *Does every polyomino that has a tiling using only translated and* 180°*-rotated copies also have such a tiling that is isohedral?*

The three-dimensional analogs of polyominoes are *polycubes*: subsets of $\mathbb{R}^3$ formed by strongly connected unions of axis-aligned unit cubes centered at locations on the cubic lattice $\mathbb{Z}^3$. Just as polyominoes tile the plane, polycubes may tile three-dimensional space.

Euler's formula implies that the average (and thus common) number of neighbors of each copy in a tiling of a polyomino is at most 6. On the other hand, polycube tilings have no such restrictions, and may tile isohedrally with arbitrarily many neighbors (see [13] for an example).

Open Problem 3 *Is there a polynomial-time algorithm for determining whether a polycube has an isohedral tiling?*

3 Non-isohedral Tilings

On the other extreme, it remains unknown whether there's an algorithm for finding any tiling at all for a given polyomino. This is a restriction of the well-known

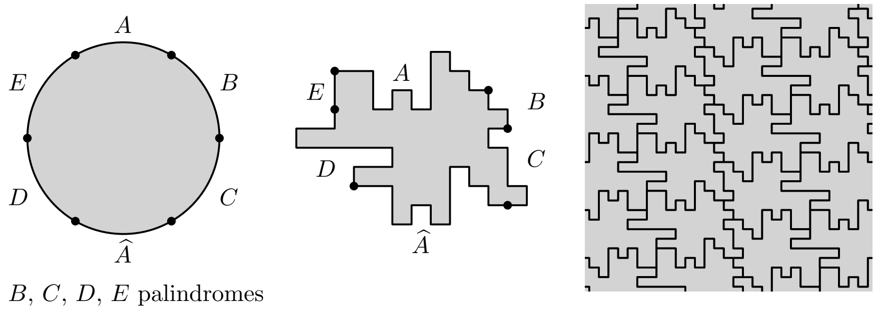

Fig. 2. Left: Conway's criterion, specified as a factorization of the polyomino's boundary. Middle: a polyomino satisfying Conway's criterion. Right: the isohedral tiling using translated and 180°-rotated copies of the polyomino induced by satisfying Conway's criterion.

open problem (e.g. appearing as Question 2.2 in [19]) regarding the undecidability of determining whether a shape has a tiling.

Open Problem 4 *Is determining whether a polyomino has a tiling decidable?*

This problem is demonstrably different than the previous; there are polyominoes that have only non-isohedral tilings (see Figure 3). Such polyominoes are called *anisohedral*.

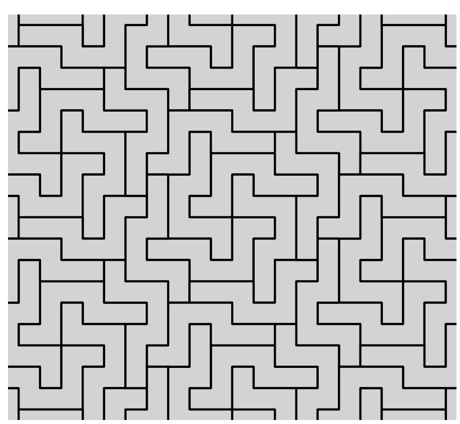

Fig. 3. A non-isohedral tiling of an anisohedral polyomino found by Rhoads [33].

Non-isohedral tilings may be partitioned into two types: *periodic* tilings with (some) symmetries, and *non-periodic* tilings with no symmetries.[1] No connected shapes that have only non-periodic tilings, called *aperiodic* shapes, are known (see [35] for more discussion).

Open Problem 5 *Is there an aperiodic polyomino?*

The next two problems concern periodic tilings. A tiling $\mathscr{T}$ is *k-isohedral* provided that k is the smallest number of partitions in a partitioning of $\mathscr{T}$ such that for any two copies T_i, T_j in a common part, there is a symmetry of $\mathscr{T}$ mapping T_i to T_j. Thus 1-isohedral is equivalent to isohedral. A polyomino is *k-anisohedral* provided it has a k-isohedral tiling, but not a k'-isohedral tiling for any $k' < k$. Examples of k-anisohedral polyominoes are known for all $k \leq 6$ [30,5] (see Figure 4), but no larger values.

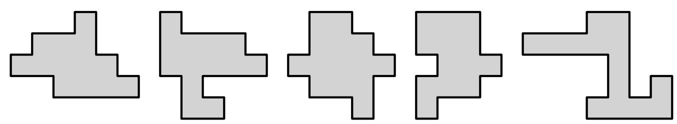

Fig. 4. Left to right: k-anisohedral polyominoes for $k = 2, 3, 4, 5, 6$ (from [30]).

Open Problem 6 *Does there exist a k-anisohedral polyomino for some $k \geq 7$?*

When attempting to determine whether a polyomino has a k-isohedral tiling, one can use a similar approach as for isohedral tilings: derive a set of sufficient boundary criteria for k-isohedral tilings, then test the polyomino against the criteria set. However, the number of such criteria grow exponentially in k, and it is unclear whether the criteria themselves could tested without incurring time proportional to the polyomino's boundary length for each criterion:

Open Problem 7 (Asked in [27]) *Is determining whether a polyomino has a k-isohedral tiling in* FPT*?*

4 Tilings by Multiple Polyominoes

Here we consider tilings that are not monohedral: the set of prototiles contains multiple polyominoes. The introduction of multiple prototiles enables creating prototile sets that only have non-periodic tilings [17]. Examples of aperiodic sets of just three polyominoes are known (see Figure 5). To the author's knowledge, no aperiodic set of two polyominoes is known:

Open Problem 8 *Does there exist an an aperiodic set of two polyominoes?*

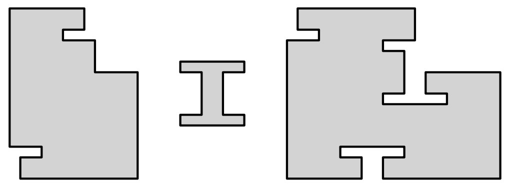

Fig. 5. An aperiodic set of three polyominoes (modified from a similar set in [1]).

Ollinger [31] proved that determining whether a set of 5 polyominoes tile the plane is undecidable, improving on a line of similar results for larger sets beginning with Berger [4] and Golomb [17]. The existence of aperiodic sets of three polyominoes opens possibility that determining whether a set of just three or four polyominoes has a tiling is undecidable:

Open Problem 9 (Open Problem 2 of [31]) *Is determining whether a set of three (or four) polyominoes tile the plane undecidable?*

5 Tilings of finite regions

The algorithmic problems of determining whether a given polyomino can tile a finite region have been shown to be NP-complete for the L-tromino and square (i.e., 2×2) tetromino [29]. As pointed out by Moore and Robson [29], these problems are closely related to the NP-hard problem of exact set cover in planar graphs, specifically bounded-size sets on grid graphs, and thus are likely NP-complete for all larger polyominoes as well. However, the author is not aware of any proof of such a result.

Open Problem 10 (Hinted at by [29]) *For every polyomino P that has a plane tiling, is determining whether P can tile a finite region* NP*-hard?*

Klarner [25] defined the *order* of a polyomino to be the minimum number of copies that can tile a rectangle. Golomb [18] proved the existence of polyominoes with order $4s$ for all $s \in \mathbb{N}$, generalizing the individual examples of polyominoes with high even orders, e.g. 76 [6] and 92 [7]. Polyominoes with other even orders are also known, with examples of orders 10 [16], 18 [25], 50 [28], and 138 [28] (see Figure 6), and polyominoes of order 2 being simple to construct. It is also known that no polyomino has order 3 [36]. This leaves a fairly wide open spectrum of possible orders which may or may not be realized by polyominoes, the smallest of which is 5:

Open Problem 11 *Is there a polyomino of order 5?*

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[1] The notion of periodic is actually more nuanced than this; see Section 3 of [19] for further discussion.

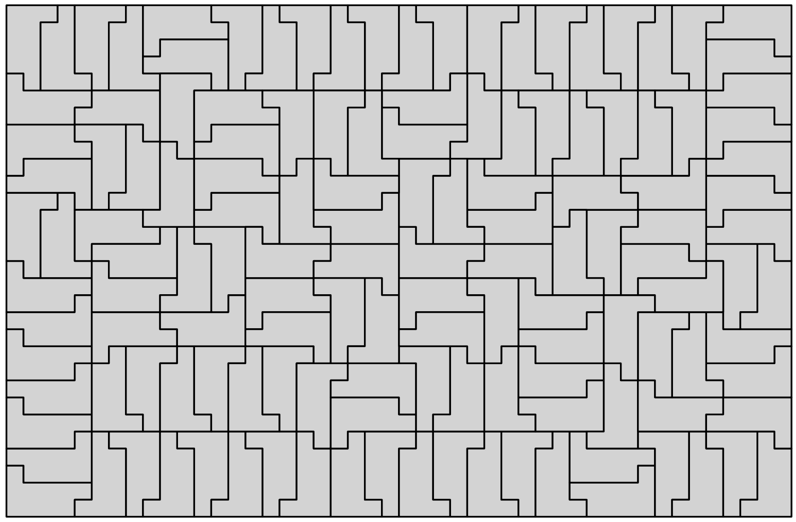

Fig. 6. The smallest rectangle tiled by the polyomino shown (found and proved in [28]). Since 138 copies of the polyomino are needed to tile the rectangle, the polyomino has order 138.

In addition to tiling rectangles, the possibility of tiling other regions with boundary can also be considered. For instance, *half-strip* regions formed by the intersection of a horizontal strip and half-plane bounded by the y-axis. Reid [32] asked whether tiling a half-strip implies that a rectangle can also be tiled:

Open Problem 12 (Question 6 of [32]) *Does every polyomino that tiles a half-strip also tile a rectangle?*

The deeply studied notion of substitution tilings [12,20] also gives rise to consideration of polyominoes that tiles scaled versions of themselves, called *rep-tiles.* Many rep-tile polyominoes are known (see Figure 7), all of which also tile rectangles. Hochberg and Reid [24] asked whether this is true of all rep-tiles:

Open Problem 13 (Asked in [24]) *Does every rep-tile polyomino tile some rectangle?*

6 Counting Tilings

It is easily observed that some polyominoes, e.g. a square, have infinitely many distinct tilings of the plane via "shifting" columns or rows by small, distinct amounts. On the other hand, some polyominoes have interlocking features that force a unique (up to symmetry) tiling. Grünbaum and Shephard [22] considered shapes inducing a finite number of tilings r, later called r*-morphic* shapes [11],

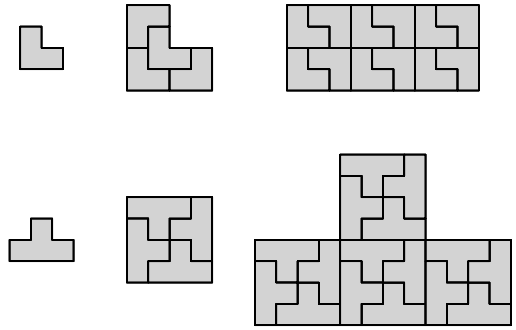

Fig. 7. Two examples of rep-tiles and their tilings of scaled versions of themselves and rectangles.

and gave examples of 2-morphic and 3-morphic polygons.[2] Fontaine and Martin later found r-morphic polyominoes for $r = 4, 5$ [9], $r = 6, 7, 8, 10$ [8], and $r = 9$ [10] (see Figure 8). However, no examples for larger r are known:

Open Problem 14 (Asked in [22]) *Does there exist an r-morphic polyomino for some $r > 10$?*

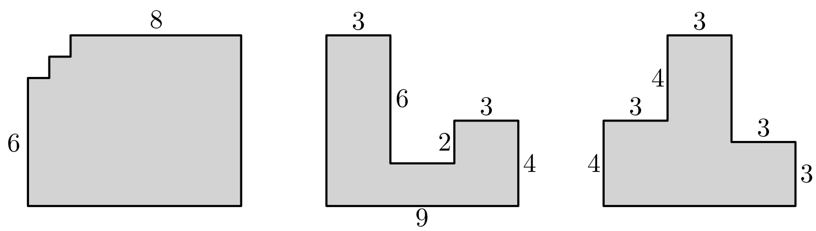

Fig. 8. From left to right: polyominoes with exactly 8, 9, and 10 distinct tilings (from [8,11]).

Following the progression in a previous section, in which the existential problem is followed by algorithmic one, how difficult is it to determine how many tilings a polyomino has?

Open Problem 15 *Is there a polynomial-time algorithm for determining whether a polyomino has r distinct tilings?*

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