Open Guard Edges and Edge Guards in Simple Polygons*

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Dedicated to Ferran Hurtado on the occasion of his 60th birthday.

Abstract. An *open edge* of a simple polygon is the set of points in the relative interior of an edge. We revisit several art gallery problems, previously considered for closed edge guards, using open edge guards. A *guard edge* of a polygon is an edge that sees every point inside the polygon. We show that every simple non-starshaped polygon admits at most one open guard edge, and give a simple new proof that it admits at most three closed guard edges. We also characterize open guard edges using a special type of kernel. Finally, we present lower bound constructions for simple polygons with n vertices that require $\lfloor n/3 \rfloor$ open edge guards, and conjecture that this bound is tight.

Keywords: art gallery, illumination, visibility, mobile guards

1 Introduction

Let P be a simply connected closed polygonal domain with n vertices. Two points $p, q \in P$ are mutually visible to each other if the closed line segment pq lies in P. In a starshaped polygon P, all points in P are visible from a single point $x \in P$, which is called a *guard point* for P. The set of all guard points is the *kernel* of P.

For a set $S \subseteq P$ of multiple guards, or the trajectories of mobile guards, we adopt the notion of weak visibility [2]. A point $p \in P$ is (weakly) visible to a set $S \subseteq P$ if it is visible from some point in S. If every point $p \in P$ is (weakly) visible from S, then S is a *guard set*.

Park *et al.* [9] considered guard sets restricted to (closed) edges of a polygon. They proved that a non-starshaped simple polygon has at most three closed guard edges, and this bound is tight. They also designed an $O(n)$ time algorithm for finding all closed guard edges in a simple n-gon. Later, it was shown that

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a shortest guard segment along the boundary of P, or anywhere in P can also be found in optimal $O(n)$ time [3,4]. A *watchman tour* is a closed curve $\gamma \subset P$ which is a guard set for P. Tan [13] gave an $O(n^5)$ time algorithm for finding a *shortest* watchman tour.

If several guards are available, we are interested in the minimum number of guards that can jointly cover any simple polygon with n vertices. By a classical result of Chvátal [5], a set of $\lfloor n/3 \rfloor$ vertex guards are always sufficient and sometimes necessary to cover a simple n-gon. It is known that $\lfloor n/4 \rfloor$ closed edge guards are sometimes necessary, and $\lfloor 3n/10 \rfloor + 1$ are always sufficient [10,11]. It is a longstanding conjecture that $\lfloor n/4 \rfloor + O(1)$ closed edge guards are always sufficient. However, $\lfloor n/4 \rfloor$ (open or closed) segment guards are always sufficient and sometimes necessary [8].

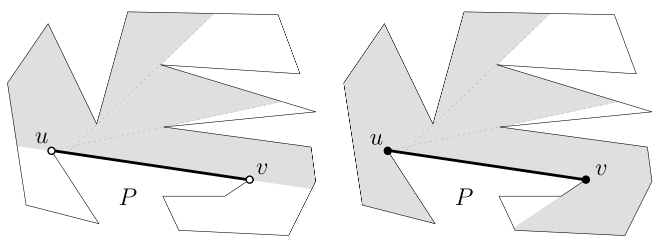

Fig. 1: The region visible by an open edge uv (left) and a closed edge uv (right) in a simple polygon.

Viglietta [14] recently suggested the use of open edge guards for various visibility problems. A *closed edge* includes the endpoints, and an *open edge* does not. See Fig. 1. Intuitively, a closed edge can "see around the corner" if its endpoint is a reflex vertex, while an open edge cannot. In this note, we examine two art gallery problems involving edges of polygons. First, *guard edges* of a polygon; single edges that guard the entire polygon. Then we consider *edge guards*; sets of edges that together guard the entire polygon.

Maximum number of guard edges in an non-starshaped simple polygon		
Guard edge type	Lower bound	Upper bound
Closed	3 [9](Section 4)	3 [9](Section 4)
Open	1 (Section 3)	1 (Section 3)

Minimum number of edge guards needed to guard any simple polygon		
Edge guard type	Lower bound	Upper bound
Closed	$\lfloor n/4 \rfloor$ [8]	$\lfloor 3n/10 \rfloor + 1$ [10,11]
Open	$\lfloor n/3 \rfloor$ (Section 6)	$\lfloor n/2 \rfloor$ (Section 6)

Table 1: A summary of new and related results.

See Table 1 for a summary of our results. We show that every non-starshaped simple polygon admits at most one open guard edge and that this bound is tight. We then use a similar technique to give a new simple proof of the result of Park et al. [9] that every non-starshaped simple polygon has at most three closed guard edges. Finally, we show that guarding some simple n-gons requires at least $\lfloor n/3 \rfloor$ open edge guards, and that no more than $\lfloor n/2 \rfloor$ such guards are ever necessary.

2 Preliminaries

It is easy to express visibility in terms of shortest paths in a simple polygon (*c.f.*, [1]). Given two points, p and q, in a simple polygon P including its boundary, the geodesic path(p, q) is the shortest directed path from p to q that lies entirely in P. Points p and q see each other iff path(p, q) is a straight line segment. Every interior vertex of path(p, q) is a reflex vertex of P. We characterize weak visibility between a point and an edge in terms of geodesics.

Lemma 1. *Let p be a point inside a simple polygon P.*

(a) *Point p is visible from an open edge uv iff p is the only common vertex of* path(p, u) *and* path(p, v)*;*
(b) *p is visible from a closed edge uv iff all common vertices of* path(p, u) *and* path(p, v) *are in $\{p, u, v\}$.*

Proof. We define a *pseudo-triangle* to be a simple polygon whose boundary consists of three reflex chains.

(a) If p is the only common vertex of the two geodesics, then uv, path(p, u), and path(p, v) form a pseudo-triangle lying in P with corners p, u and v. Each corner of a pseudo-triangle is weakly visible from the opposite side, hence p is visible from a point in uv (Fig. 2, left). If $q \neq p$ is the last vertex in common on the two geodesics path(p, u) and path(p, v), then q is an interior point of every geodesic from p to any $w \in uv$. Hence p is not visible from any point of the open edge uv (Fig. 2, middle).

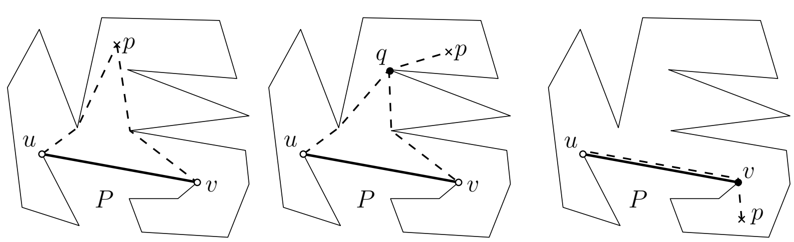

Fig. 2: The geodesics path(p, u) and path(p, v). Left: p is the only common vertex of path(p, u) and path(p, v). Middle: the common vertices are p and q. Right: The common vertices are p and v.

(b) If p is the only common vertex of the two geodesics, then p is visible from an interior point of uv as in part (a). If u or v is the only common vertex (apart from p) of the two geodesics, then point p is directly visible from u or v (Fig. 2, right). Finally, if $q \notin \{p, u, v\}$ is a common vertex of the two geodesics, then q is an interior point of every geodesic from p to any $w \in uv$, and hence p is not visible from any point of the closed edge uv. □

3 Open Guard Edges

In this section we consider open guard edges. Observe that every edge of a convex polygon is a guard edge, since it lies in the kernel of the polygon; but there may be $n/4$ or more open guard edges even if all edges are disjoint from the kernel (Fig. 3, left). In this section, we show that every non-starshaped simple polygon has at most one open guard edge. This bound is tight, as shown by the example in Fig. 3, right.

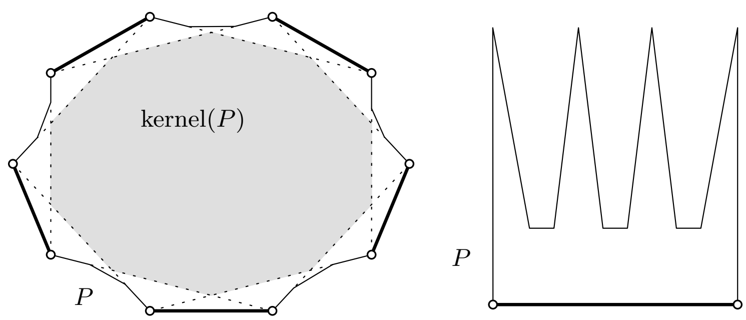

Fig. 3: Left: a starshaped n-gon P with $n/4$ open guard edges where the kernel lies in the interior of P. Right: a non-starshaped polygon with one open guard edge.

We prove the upper bound by contradiction: we prove that a simple polygon with at least two open guard edges is starshaped. Let P be a simple polygon, and suppose that edges ab and cd are open guard edges. We may assume without loss of generality that a, b, c, d are in counterclockwise order along the boundary of P (possibly, $b = c$ or $d = a$).

Lemma 2. path(b, c) *and* path(a, d) *are disjoint.* path$(a, c) = ac$ *and* path$(b, d) = bd$ *are line segments.*

Proof. Note that ab, path(b, c), cd, and path(a, d) form a geodesic quadrilateral Q, i.e. a quadrilateral where each side is a geodesic path. Every geodesic between a point in ab and a point in cd lies in Q. If path(b, c) and path(a, d) have a common interior vertex q, then a or b is not visible from the open edge cd by Lemma 1, and so cd cannot be a guard edge. We conclude that path(b, c) and path(a, d) are disjoint, and Q is a simple polygon.

The geodesics path(a, c) and path(b, d) lie in Q, as otherwise path(b, c) or path(a, d) is not a geodesic. So any interior vertex of path(a, c) and path(b, d) is a vertex of Q. If an interior vertex of path(a, c) is in path(b, c), then c is not visible from ab. Similarly, if an interior vertex of path(a, c) is in path(a, d), then a is not visible from cd. Hence, path(a, c) has no interior vertices. Analogous argument shows that path(b, d) has no interior vertices, either. □

Lemma 3. *The intersection point $x = ac \cap bd$ is in the kernel of P.*

Proof. Refer to Fig. 4. It is enough to show that an arbitrary point p in polygon

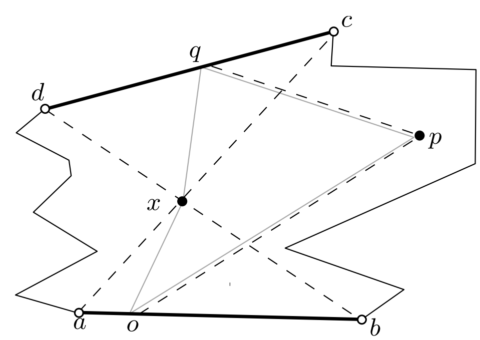

Fig. 4: A schematic of the proof that a simple polygon with two open guard edges must be starshaped. The guard edges are ab and cd. The point $x = ac \cap bd$ is in the kernel of the polygon, since every point $p \in P$ is visible from x.

P is visible from x. By Lemma 2, ac and bd are diagonals of P. The triangles $\Delta(abx)$ and $\Delta(cdx)$ lie inside P. If $p \in \Delta(abx)$ or $p \in \Delta(cdx)$, then segment px lies in the same triangle.

Assume now that p is outside of both triangles. Since ab and cd are open guard edges, p sees some points in their relative interiors, say $o \in ab$ and $q \in cd$. So all the edges of the quadrilateral $Q = (o, p, q, x)$ lie in P, i.e. the boundary of Q lies in P. Thus any holes in Q are also holes in P, and since P is simple, Q must also be simple and lie in P. Since the vertices o and q of Q are interior points of the edges ab and cd, they are convex vertices in Q. So the diagonal px of Q lies in the interior of Q and P. □

Theorem 1. *Every non-starshaped simple polygon has at most one open guard edge.*

Proof. If a simple polygon has two open guard edges, then it has a nonempty kernel by Lemma 3, and thus is starshaped. So every non-starshaped simple polygon has at most one open guard edge. □

Remark. The upper bound of Theorem 1 does not apply to polygons with holes. Note that an open edge on the boundary of a hole cannot see the entire boundary of the hole. So all open edge guards are on the boundary of the outer polygon. By the result in [9] there are at most 3 *closed* guard edges on the outer boundary of a polygon with holes. Since every open guard edge is a closed guard edge, as well, a polygon with holes has at most 3 open guard edges. This upper bound is tight, as shown by the following simple construction. Let the outer polygon and a hole be two centrally dilated triangles. Then all three open edges of the outer polygon are guard edges.

4 Closed Guard Edges

In this section, we extend the argument of the previous section to give a short proof for the following result of Park *et al.* [9].

Theorem 2 ([9]). *Every non-starshaped simple polygon has at most three closed guard edges.*

We proceed by contradiction, and show that the presence of four closed guard edges implies that the polygon is starshaped. Let P be a simple polygon where g_1, g_2, g_3, and g_4, in counterclockwise order, are guard edges. Let $g_1 = ab$ and $g_3 = cd$ such that a, b, c, and d are in counterclockwise order along P. Note that the vertices a, b, c, and d are distinct.

Lemma 4. *The geodesics* $\mathrm{path}(b,c)$ *and* $\mathrm{path}(a,d)$ *are disjoint; and all vertices of the geodesics* $\mathrm{path}(a,c)$ *and* $\mathrm{path}(b,d)$ *are in* $\{a,b,c,d\}$.

Proof. Consider the geodesic quadrilateral Q formed by ab, $\mathrm{path}(b,c)$, cd, and $\mathrm{path}(a,d)$. Every geodesic between a point in ab and a point in cd lies in Q. Suppose that an interior vertex q of $\mathrm{path}(b,c)$ is a vertex of $\mathrm{path}(a,d)$. If $q = a$ or an interior vertex of $\mathrm{path}(a,d)$, then b is not visible from the closed edge cd by Lemma 1. Similarly, if $q = d$, then c is not visible from the closed edge ab. We conclude that $\mathrm{path}(b,c)$ and $\mathrm{path}(d,a)$ are disjoint, and Q is a simple polygon.

The geodesics $\mathrm{path}(a,c)$ and $\mathrm{path}(b,d)$ lie in Q, so any interior vertex of $\mathrm{path}(a,c)$ and $\mathrm{path}(b,d)$ is a vertex of Q. If $\mathrm{path}(a,c)$ and $\mathrm{path}(b,c)$ have a common interior vertex, then c is not visible from ab. Similarly, no two geodesics from $\{a,b\}$ to $\{c,d\}$ can have any common interior vertex. Hence all interior vertices of $\mathrm{path}(a,c)$ and $\mathrm{path}(b,d)$ are in $\{a,b,c,d\}$. □

Corollary 1.
- *If* $\{a,b,c,d\}$ *is in convex position, then* $\mathrm{path}(a,c) = ac$ *and* $\mathrm{path}(b,d) = bd$. *Fig. 5, left.*
- *Otherwise suppose w.l.o.g. that* $\mathrm{conv}(\{a,b,c,d\}) = \Delta(abc)$. *Then* $\mathrm{path}(a,c) = (a,d,c)$ *and* $\mathrm{path}(b,d) = bd$. *Fig. 5, right.*

Lemma 5. *The intersection point* $x = \mathrm{path}(a,c) \cap \mathrm{path}(b,d)$ *is in the kernel of* P.

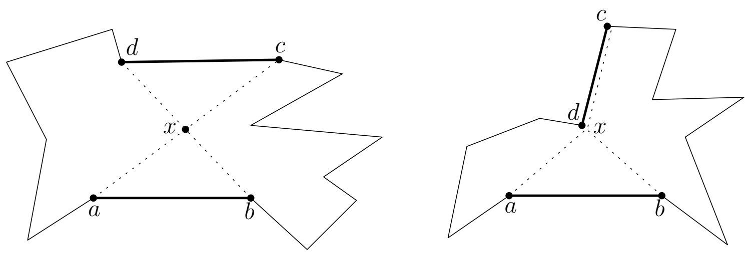

Fig. 5: The convex hull of two closed guard edges, ab and cd, is either a quadrilateral or a triangle.

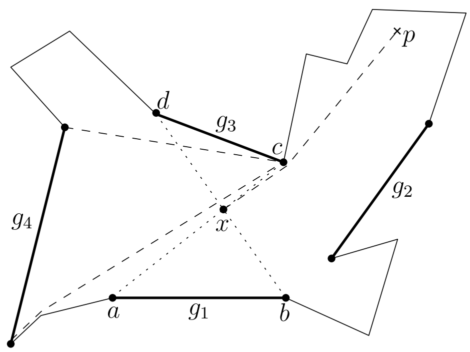

Fig. 6: A schematic of the proof that a simple polygon with four closed guard edges must be starshaped. Suppose that $g_1 = ab$, g_2, $g_3 = cd$, and g_4 are guard edges. If a point $p \in P$ is not visible from $x = \text{path}(a, c) \cap \text{path}(b, d)$, then we show that p is also not visible from g_2 or g_4.

Proof. It is enough to show that an arbitrary point p in polygon P is visible from x. By Corollary 1, the triangles $\Delta(abx)$ and $\Delta(cdx)$ lie inside P (one of the triangles may be degenerate). If p is in $\Delta(abx)$ or $\Delta(cdx)$, then segment px lies in the same triangle. Refer to Fig. 6.

Assume now that p is outside of both triangles and, w.l.o.g. it is on the right side of the directed geodesics $\text{path}(a, c)$ and $\text{path}(b, d)$. That is, p and the guard edge g_4 are on opposite sides of these geodesics.

If $\text{path}(p, x) = px$, then p is visible from x. Suppose, to the contrary, that $\text{path}(p, x)$ is not a straight line segment. Assume w.l.o.g. that $\text{path}(p, x)$ makes a *right* turn at its last interior vertex q. Then $\text{path}(p, d)$ also makes a right turn at q. Since p is visible from the guard edge cd, we must have $q = c$ by Lemma 1(b). Recall that any geodesic from p to a point in g_4 crosses both $\text{path}(a, c)$ and $\text{path}(b, d)$. Since we assumed that $\text{path}(p, x)$ makes a right turn at c, every geodesic from p to a point in g_4 also makes a right turn at c. However, c is disjoint from g_4, and by Lemma 1(b), p is not visible from g_4, contradicting our initial assumption. We conclude that $\text{path}(p, x)$ is a straight line segment, and so p is visible from x. $\square$

Proof of Theorem 2. If a simple polygon has four closed guard edges, then it has a nonempty kernel by Lemma 5, and thus is starshaped. So every non-starshaped simple polygon has at most three closed guard edges. □

5 Characterizing Open Guard Edges

In this section, we characterize the open guard edges of a simple polygon P in terms of the left and right kernels of P (defined below).

5.1 Left and right kernels

Recall that the set of points from which the entire polygon P is visible is the *kernel*, denoted $K(P)$, which is the intersection of all closed halfplanes bounded by a supporting line of an edge of P and facing towards the interior of P. Lee and Preparata [6] designed an optimal $O(n)$ time algorithm for computing the kernel of simple polygon with n vertices. We now define a weaker version of the kernel: the *left* and *right kernels* of P, denoted $K_{\text{left}}(P)$ and $K_{\text{right}}(P)$.

For every reflex vertex r, we define two polygons $C_{\text{left}}(r) \subset P$ and $C_{\text{right}}(r) \subset P$. Shoot a ray from r in a direction collinear with the edge incident to r preceding (resp., following) r in counterclockwise order; and let $C_{\text{left}}(r)$ (resp., $C_{\text{right}}(r)$) be the part of P on the left (resp., right) of the ray. These polygons have previously been defined in [3]. It is clear that if P is weakly visible from a set $S \subset P$, then S must intersect both $C_{\text{left}}(r)$ and $C_{\text{right}}(r)$ for every reflex vertex r.

Now we define $K_{\text{left}}(P)$ as the intersection of polygons $C_{\text{left}}(r)$ for all reflex vertices r; and $K_{\text{right}}(P)$ as the intersection of polygons $C_{\text{right}}(r)$ for all r. See Fig. 7 for an example. Clearly, we have

$$K(P) = K_{\text{left}}(P) \cap K_{\text{right}}(P).$$

By construction, both $K_{\text{left}}(P)$ and $K_{\text{right}}(P)$ are convex polygons, whose edges are collinear with some of the edges of P.

5.2 Left and right kernel decompositions

In the following lemma we use two decompositions of a polygon based on its left and right kernels. The *left kernel decomposition* is the decomposition of the polygonal domain P in the following way: One cell of the decomposition is the left kernel $K_{\text{left}}(P)$. The region inside P but in the exterior of $K_{\text{left}}(P)$ is decomposed by extending each edge of $K_{\text{left}}(P)$ in clockwise direction. Refer to Fig. 7. Since $K_{\text{left}}(P)$ lies on the left side of rays emitted from reflex vertices of P, the clockwise extension of every edge of $K_{\text{left}}(P)$ reaches a collinear edge of P. The right kernel decomposition is defined analogously: one cell is $K_{\text{right}}(P)$, and the rest of P is decomposed by counter-clockwise extensions of the edges of $K_{\text{right}}(P)$. Note that if an open edge of P is disjoint from the left kernel, then it is adjacent to a unique region of the left kernel decomposition. Additionally, each region of the decomposition, except for $K_{\text{left}}(P)$, has exactly one common edge with the left kernel.

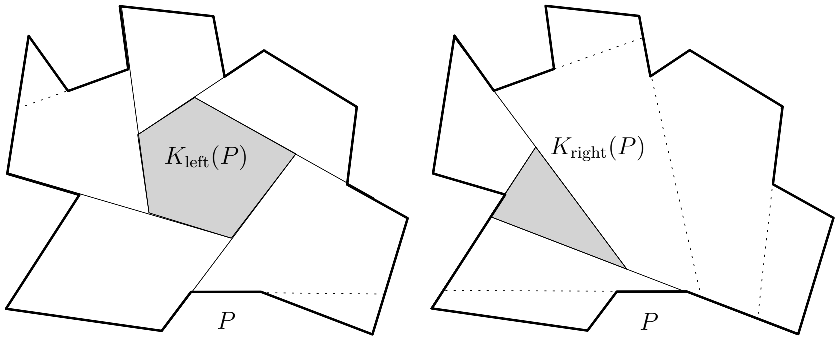

Fig. 7: The left and right kernels of a polygon. The dotted lines bound some polygons $C_{\text{left}}(r)$ and $C_{\text{right}}(r)$, but they are not part of the kernel decompositions.

Lemma 6. *An open edge e of a simple polygon P is a guard edge of P iff e intersects both the left and the right kernels of P.*

Proof. Let $e = uv$ be an open edge of P. First assume that e is disjoint from the left kernel $K_{\text{left}}(P)$. Then e is adjacent to a unique region in the left kernel decomposition of P. This region is adjacent to a unique edge k of $K_{\text{left}}(P)$, and k lies on a ray emitted by a reflex vertex r on P. Then e and the polygon $C_{\text{left}}(r)$ lies on opposite sides of the supporting line of k. Hence e does not intersect $C_{\text{left}}(r)$, and so it is not a guard edge.

Now assume that $e = uv$ is not a guard edge, that is, there is a point $p \in P$ such that p is not visible from e. By Lemma 1(a), the geodesics $\text{path}(p, u)$ and $\text{path}(p, v)$ have common interior vertices. Let r be their last common vertex, which is necessarily a reflex vertex of P, and assume w.l.o.g. that both geodesics make a right turn at r. Then $p \in C_{\text{left}}(r)$, but e is disjoint from $C_{\text{left}}(r)$. That is, e is disjoint from the left kernel of P. □

6 Open Edge Guards

Recall that every simple polygon with n vertices can be covered by $\lfloor 3n/10 \rfloor + 1$ closed edge guards, and there are n-gons that require at least $\lfloor n/4 \rfloor$ closed edge guards. It turns out that the endpoints of the closed edge guards are crucial for these bounds. Significantly more edge guards may be necessary if the endpoints are removed.

We construct four different infinite families of polygons that require $\lfloor n/3 \rfloor$ open edge guards for n vertices. Refer to Fig. 8. The lower bounds for all our constructions can be verified by a standard "hidden point" argument. We hide $\lfloor n/3 \rfloor$ points (gray dots in Fig. 8) in the interior of a polygon such that each open edge guard sees exactly one such point. That is, each hidden point requires a unique open edge guard, and any set of fewer than $\lfloor n/3 \rfloor$ open edge guards would miss at least one hidden point.

It is not difficult to see that $\lfloor n/2 \rfloor$ open edge guards are always sufficient. Partition the set of edges of the polygon into two subsets for which the interior normals of the edges have either a positive or negative y-component. Each subset of open edges jointly covers the entire polygon. We conjecture this upper bound is weak, and that $\lfloor n/3 \rfloor$ is the tight bound for the number of open edge guards necessary and sufficient to guard any simple polygon with n vertices.

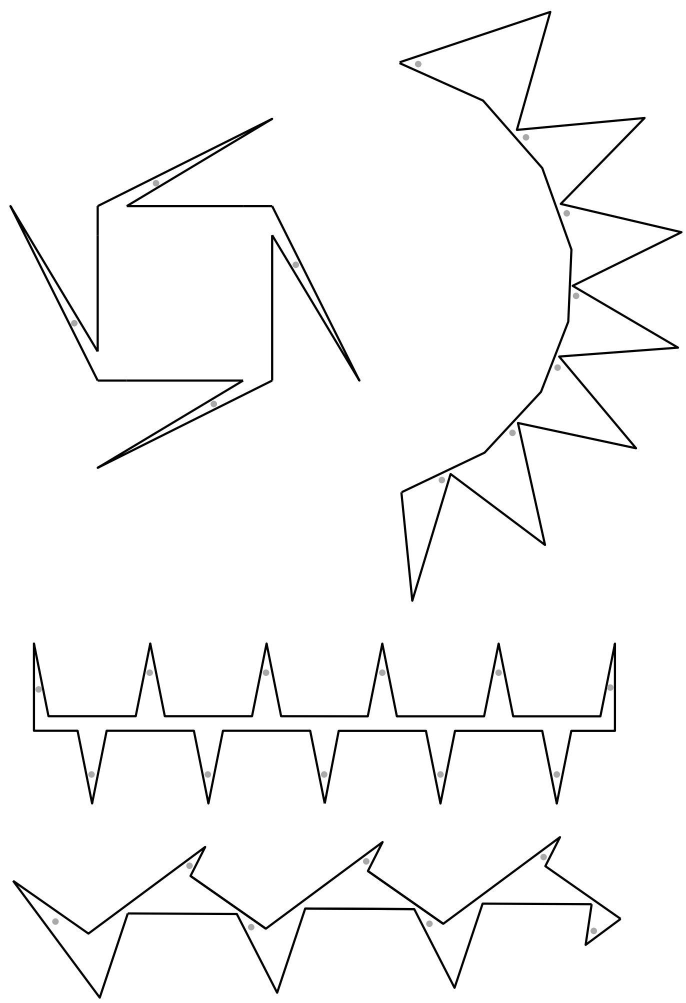

Fig. 8: Examples of polygons requiring $n/3$ open edge guards. The gray dots in each polygon indicate a set of points that require a distinct edge to guard each point.

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