Packing Cube Nets into Rectangles with $O(1)$ Holes

Erik D. Demaine[1], Martin L. Demaine[1], Ryuhei Uehara[2], Yushi Uno[3], and Andrew Winslow[4]

[1] Massachusetts Institute of Technology, USA
edemaine@mit.edu, mdemaine@mit.edu
[2] Japan Advanced Institute of Science and Technology, Japan
uehara@jaist.ac.jp
[3] Osaka Prefecture University, Japan
uno@cs.osakafu-u.ac.jp
[4] San Bruno, CA, USA
andrewwinslow@gmail.com

Abstract. We show that the 11 hexomino nets of the unit cube (each used an unlimited number of times) can pack disjointly into an $m \times n$ rectangle and cover all but a constant c number of unit squares, where $4 \leq c \leq 14$ for *all* integers $m, n \geq 2$. On the other hand, the nets of the dicube (two unit cubes) can be exactly packed into some rectangles.

Keywords: unfoldings · packings · polyominoes · hexominoes

1 Introduction

Packing with polyominoes. Polyominoes[5] have attracted the attention of mathematicians, computer scientists, and amateur researchers since their invention by Solomon Golomb in 1954 [9,11] and popularization by Martin Gardner since 1959 [7,8]. Since this early beginning, the primary focus has been to study packing of polyominoes into "nice" shapes. For example, Golomb's original paper [9] studies which subsets of the 8×8 checkerboard can and cannot be exactly packed with dominoes, trominoes, and/or tetrominoes, possibly with a small number of monominoes. Gardner's original article [7] includes a proof by Golomb that the 35 hexominoes (each used exactly once) cannot exactly pack any rectangle, even though there are six rectangles with the proper area. Gardner wrote: "I seriously considered offering \$1,000 to the first reader who succeeded in constructing one of these six rectangles, but the appalling thought of hours that might be wasted on the challenge forced me to relent." In this spirit, the Eternity puzzle is a 209-piece polyomino-like packing puzzle that awarded £1 million to the first solvers, Alex Selby and Oliver Riordan, one year after its release [16,20].

[5] An *n-omino* or *polyomino* is an edge-to-edge joining of n unit squares. The special cases $n = 1, 2, 3, 4, 5, 6$ are called *monominoes*, *dominoes*, *trominoes*, *tetrominoes*, *pentominoes*, and *hexominoes*, respectively.

Packing with hexominoes. In this paper, we study packing of *hexominoes* into *rectangles*. In 1966, Golomb [10] proved that nine hexomino shapes can each (used an unlimited number of times) exactly pack some finite rectangle, while 25 hexomino shapes can each exactly pack infinite forms of rectangles (three-sided half-strips, four-sided Ls, two-sided strips, or the entire plane) but cannot exactly pack any finite rectangle. As described by Gardner [8], these results left unsolved exactly one hexomino shape (of 35); 21 years later, blind software engineer Dahlke [5] used computer search to find an exact packing of this hexomino shape into the smallest possible rectangle, 23×24. Every exact packing of hexominoes into some finite rectangle leads to exact packings into infinitely many finite rectangles (by repetition). Klarner [14] studied which rectangles can be exactly packed by a single polyomino shape. Bos [2] gave several exact packings of squares by two hexomino shapes (each used an unlimited number of times). See the extensive bibliography on "rectifiable polyominoes" [18]. In the modern Internet era, many websites are devoted to exact packings of nice shapes by various sets of hexominoes [3,4,6,17,18,21].

Cube nets. We focus on the eleven hexominoes that fold into a unit cube, as shown in Fig. 1. More precisely, an *edge unfolding* of a cube is a way to cut some edges and unfold the remaining edges to produce a flat nonoverlapping polygon, called a *net*. Only eleven of the 35 hexominoes are nets of the cube. For convenience, we name them alphabetically from A to K (different from the usual labeling of hexominoes).

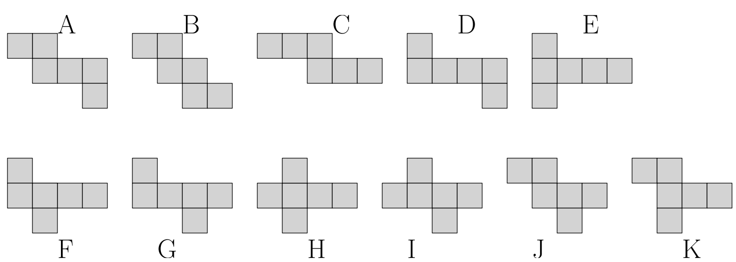

Fig. 1. Eleven cube nets labeled from A to K (as in [12]).

What about packings by cube nets? A recent paper [1] considered exact packing of cube nets (each used an unlimited number of times) onto the surface of a (larger) cube. For packing into rectangles, none of the eleven cube nets are among the ten hexomino shapes that by themselves can exactly pack a rectangle. More surprising is that no combination of cube nets can exactly pack a rectangle:

Theorem 1 (Folklore, proved in [12,13]). *No rectangle can be exactly packed by nets of the cube.*

Near-exact packing. Therefore, we relax the goal, and aim instead for "near-exact" packings of cube nets into various rectangles. In other words, we aim to find packings that have no overlap and cover all but a few 1×1 *holes* or *uncovered cells*. Specifically, we give nearly tight bounds on the following problem:

> For all positive integers m, n, what is the maximum number of cube nets that can be packed into an $m \times n$ rectangle, or equivalently, what is the fewest uncovered cells that can remain?

This specific problem has been studied extensively for small m, n. It is known, as folklore, that the minimum-area rectangle into which all 11 nets can pack is $11 \times 7 = 77$. In perhaps the first work on the problem, Odawara [15] presents several packings and gives a table of results that he could achieve up to 12×12. Most intriguingly, he conjectured that every rectangle with $m, n \geq 7$ can have all but between 6 and 11 squares covered by packing cube nets. Inoha et al. [12] improved and extended these results by exhaustive computer search using BurrTools [19]. Table 1 summarizes all of the results now known to be optimal.

2	4			
3	6	9		
4	8	6	4	
5	4	9	8	7
×	2	3	4	5

6	6	6	6	6	12							
7	8	9	4	11	6	7						
8	4	6	8	10	6	8	10					
9	6	9	6	9	6	9	6	9				
10	8	6	4	8	6	10	8	6	10			
11	4	9	8	7	6	11	10	9	8	7		
12	6	6	6	6	12	6	6	6	6	6	6	
13	8	9	4	11	6	7	8	9	10	11	6	7
14	4	6	8	10	6	8	10	6	8	10	6	8
×	2	3	4	5	6	7	8	9	10	11	12	13

Table 1. Known optimal values of uncovered cells in each size $m \times n$ of rectangles for $2 \leq n \leq m \leq 14$. The value is 12 only for rectangles 6×6 and 12×6.

Our results. Our main result is a proof of a slightly weaker conjecture: every $m \times n$ rectangle with $m, n \geq 2$ can be packed by cube nets leaving at most 14 uncovered cells (Section 2). ($1 \times n$ rectangles cannot fit any cube net from Fig. 1, so these are all rectangles of interest.) Such a worst-case upper bound cannot be improved beyond 12, as the 6×6 and 6×12 rectangles require that many holes, so our upper bound of 14 is close to tight. We also prove a stronger form of Theorem 1: every $m \times n$ rectangle with $m, n \geq 2$ must leave at least 4 uncovered cells (Section 3). This best-case lower bound is tight because 2×8, 4×7, etc. rectangles can be packed with just 4 uncovered cells. By contrast, we show that the nets of a dicube (two cubes glued face-to-face) behave very differently: they admit an exact packing of some rectangle, and thus infinitely many rectangles (Section 4).

BurrTools. BurrTools [19] is a powerful software tool for exhaustively exploring packings of a specified set of shapes into a specified shape (e.g., a rectangle).

Fig. 2 shows a snapshot of the software. We used BurrTools extensively on constant-size instances to search for patterns in the packings which we then generalized by hand into the infinite family of packings presented in Section 2.

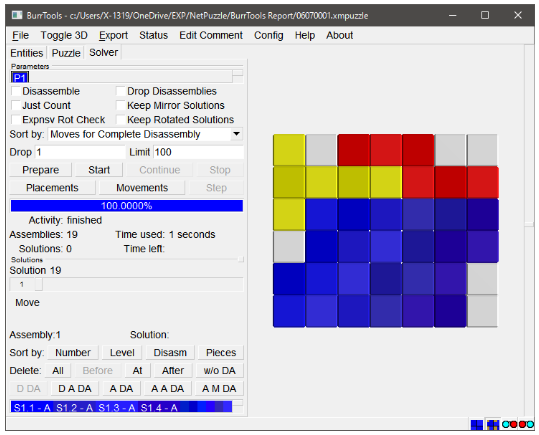

Fig. 2. A snapshot of BurrTools.

2 Upper Bound on Uncovered Cells

In this section, we prove the following theorem, which upper bounds the worst-case number of uncovered cells when we pack nets of the cube in a rectangle.

Theorem 2. *Let $m, n \in \mathbb{N}$. If $m \geq n \geq 2$, then nets of the cube can pack an $m \times n$ rectangle leaving at most 14 uncovered cells.*

We prove this theorem by constructing packing patterns that satisfy the stated condition. We split into four cases according to the size $m \times n$ of the rectangle: (i) $\{\geq 8\} \times \{\geq 6\}$ (i.e., 8×6 or larger); (ii) $\{6, 7\} \times \{6, 7\}$; (iii) $\{\geq 6\} \times \{2, 3, 4, 5\}$; (iv) $\{2, 3, 4, 5\} \times \{2, 3, 4, 5\}$ (i.e., 5×5 or smaller). Each of these cases corresponds to the subsequent lemmas.

Lemma 1. *Rectangles of dimensions $m \times n$ with $m \geq 8$ and $n \geq 6$ (and $m \geq n$) can be packed with nets of the cube leaving at most 14 cells uncovered.*

Proof. The proof is constructive, broken into several cases based around small modifications to two general packing format seen in Fig. 3. The two top-level cases are rectangles of even ($6i + i'$ with $i' \in \{0, 2, 4\}$) and odd ($6i + i'$ with $i' \in \{1, 3, 5\}$) width. Notice that this combination can realize any width of 6 or larger. We also remark that these figures illustrates how we start packing the top

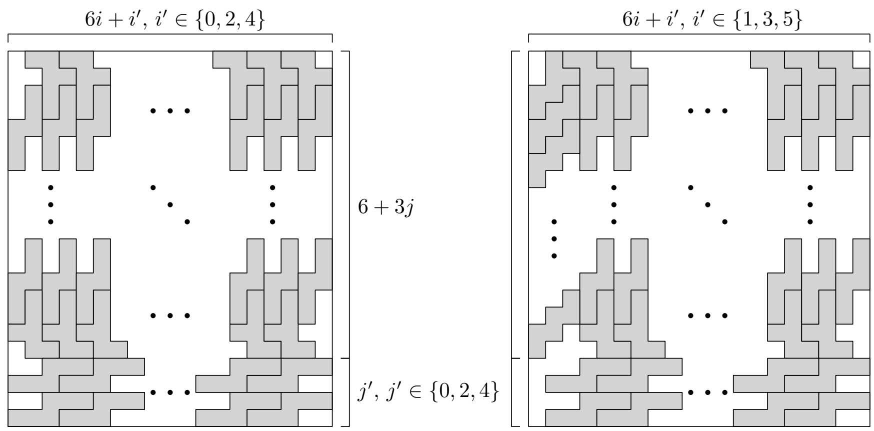

Fig. 3. The approach to packing rectangles with nets of the cube.

part of a rectangle. Each case is further broken into subcases based on height $6 + 3j + j'$ with j' equal to 0, 2, or 4.

Each subcase (combination of i' and j' values) is considered separately. For compactness, the arrangement of the upper portion of each packing is excluded unless it does not agree with the arrangement seen in Fig. 3. Figs. 4 and 5 contain the subcases with $i' \in \{0, 2, 4\}$ and $i' \in \{1, 3, 5\}$, respectively.

The vertical dimension is assumed to be equal to $6 + 3j + j'$ with $j \in \{0, 2, 4, \ldots\}$ and $j' \in \{0, 2, 4\}$. Notice again that any integer at least 8 can be written in such a form. Inspection of the arrangement of each subcase is sufficient to observe that every subcase accommodates all values of $i \geq 1$ (i.e. widths of $6i + i'$ for some subset of i' and all $i \geq 1$).

In the even-width case, we first see that there are 6 cells remain uncovered in the top region (Fig. 3, left), which implies that if suffices to show that we can cover the other (especially the bottom) region with at most 8 cells remain uncovered. We can confirm this fact in Fig. 4, except the case that the width is $6i+i'$ with $i' = 4$ and the height is $6+3j+j'$ with $j' = 4$. This case is considered separately, and is covered with 10 uncovered cells in total (Fig. 4, bottom right).

In the odd-width case, we see similarly that there are 3 cells remain uncovered in the top region (Fig. 3, right), which implies that if suffices to show that we can cover the other region with at most 11 cells remain uncovered. We can confirm all the cases in Fig. 5. This completes the proof. □

Lemma 2. *Rectangles of dimensions* $m \times n$ *with* $7 \geq m \geq n \geq 6$ *can be packed with nets of the cube leaving at most 12 cells uncovered.*

Proof. Fig. 6 shows such packings found using BurrTools. □

Lemma 3. *Rectangles of dimensions* $m \times n$ *with* $m \geq 6$ *and* $5 \geq n \geq 2$ *can be packed with nets of the cube leaving at most 12 cells uncovered.*

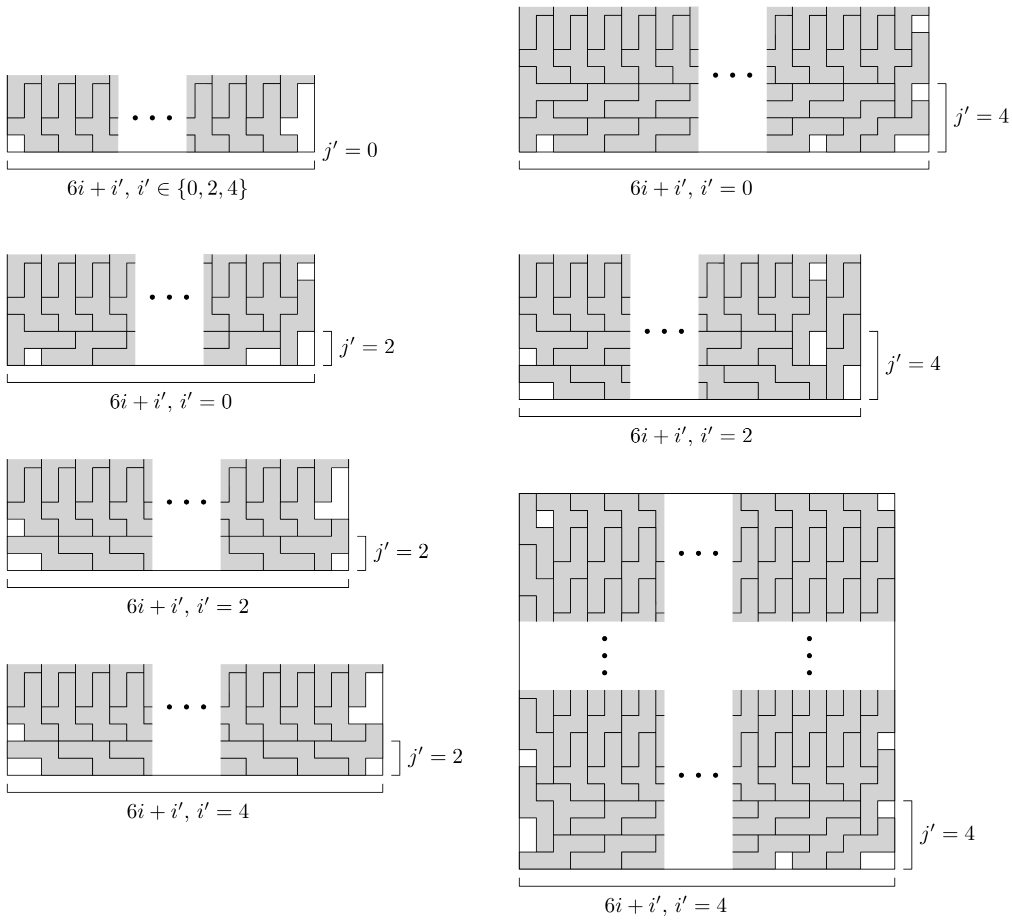

Fig. 4. The subcases of packing even-width rectangles with nets of the cube.

Proof. Fig. 7 shows packings for each subcase of $m \geq 6$. In each subcase, at most 5 cells are uncovered at the right end of the rectangle, and at most 7 cells are left uncovered at the left end of the rectangle, for a total of at most 12 (< 14) cells uncovered. □

Lemma 4. *Rectangles of dimensions $m \times n$ with $5 \geq m \geq n \geq 2$ can be packed with nets of the cube leaving at most 10 cells uncovered.*

Proof. The cases $\{2, 3, 4, 5\} \times 2$ and $3 \times \{2, 3\}$ have area ≤ 10, so the empty packing suffices. The remaining cases are $4 \times \{3, 4\}$ and $5 \times \{3, 4, 5\}$. Refer to Fig. 1. Any single net other than C fits in a 4×3 rectangle and occupies 6 squares, which is a sufficient packing for $4 \times \{3, 4\}$ (leaving 6 and 10 cells uncovered, respectively) and for 5×3 (leaving 9 cells uncovered). Case 5×4 can be solved

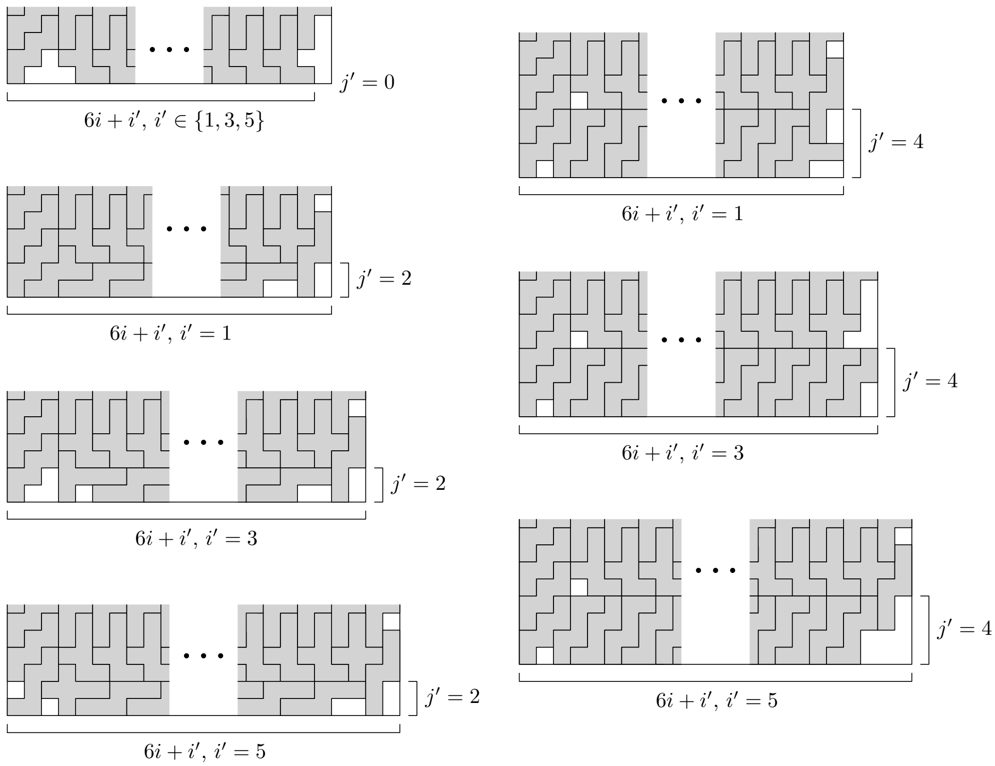

Fig. 5. The subcases of packing odd-width rectangles with nets of the cube.

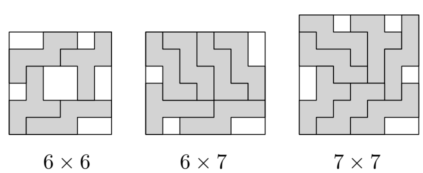

Fig. 6. Packings of 6×6, 6×7, and 7×7 rectangles by cube nets leaving 12, 6, and 7 uncovered cells, respectively.

by stacking two C nets vertically (leaving 8 cells uncovered). Case 5×5 can be solved by stacking a C above a D (or A) above another C (leaving 7 cells uncovered). □

The previous four lemmas together establish Theorem 2, that 14 is an upper bound on the number of uncovered cells for $m \times n$ with $m \geq n \geq 2$.

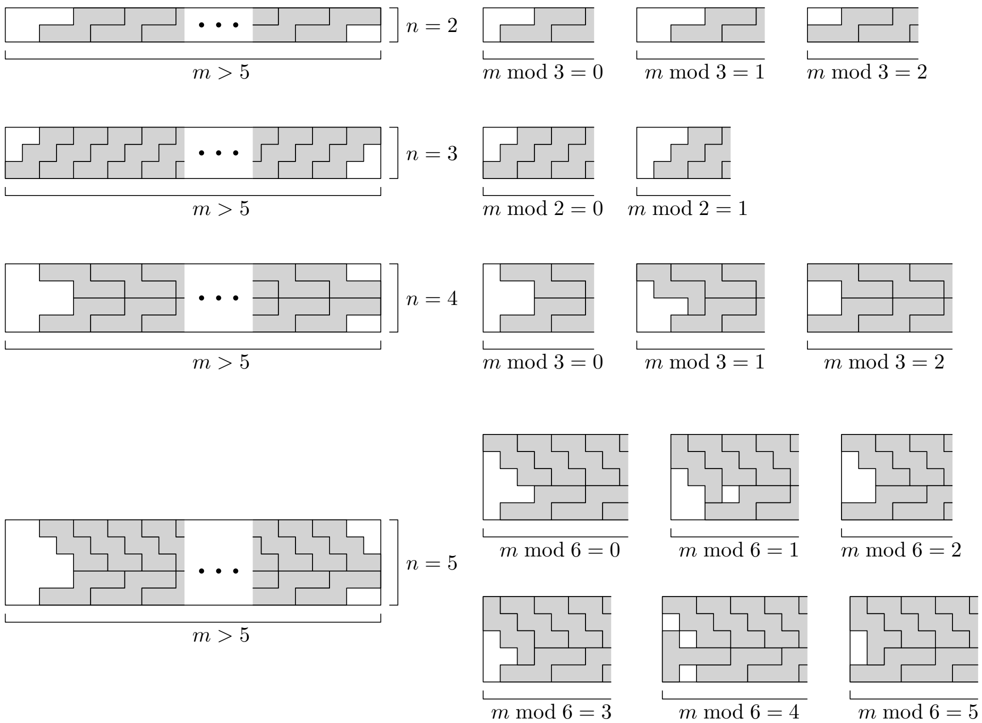

Fig. 7. The subcases of packing rectangles of dimensions $m \geq 6$ and $5 \geq n \geq 2$ with nets of the cube.

3 Lower Bound on Uncovered Cells

In this section, we show the following theorem, which tells us the best case number of uncovered cells when we pack nets of the cube in a rectangle. We cannot do better than leaving 4 uncovered cells for a rectangle of any dimension.

Theorem 3. *Let $m, n \in \mathbb{N}$. If $\{m, n\} \notin \{\{1, 1\}, \{1, 2\}, \{1, 3\}\}$, then any packing of nets of the cube into an $m \times n$ rectangle leaves at least 4 uncovered cells.*

We prove this theorem by dividing it into the following two cases by the sizes of rectangles, that is, (i) 6×2, or larger, and (ii) 5×5, or smaller. Each of these corresponds to the subsequent lemmas. Subsequent discussions build on ideas in [12,13].

Lemma 5. *For rectangles of dimensions $m \times n$ with $m \geq 6$ and $n \geq 2$ $(m \geq n)$, any packing of nets of the cube leaves at least 4 uncovered cells.*

Proof. Each of four corners of an empty rectangle is an uncovered cell. We first notice that nets H and I cannot cover a corner cell without protrusion. Observe that any covering of this cell via all possible placements of nets of the cube (seen

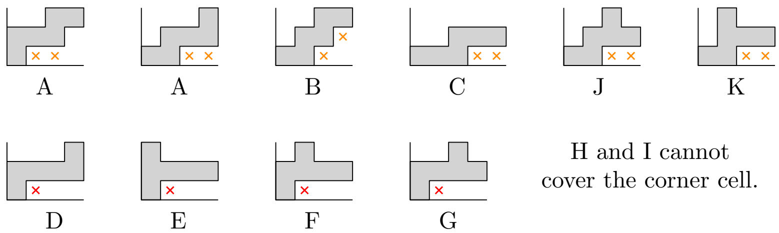

Fig. 8. All possible coverings of the corner of a rectangle by nets of the cube.

in Fig. 8, up to symmetry) leaves new uncovered cells that are either *uncoverable* (red X's in Fig. 8) or *problematic* (orange X's in Fig. 8).

Placements that do not create uncoverable cells leave problematic cells in only two arrangements, seen in Fig. 9. We call these arrangements A/C/J/K and B, after the two sets of nets of the cube that create them.

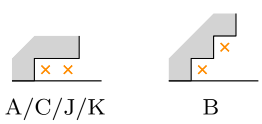

Fig. 9. The two cases of remaining problematic cells after covering the corner of a rectangle by a net of the cube.

For the two problematic cell arrangements A/C/J/K and B in Fig. 9, only four net placements suffice to cover both problematic cells, seen in Fig. 10.

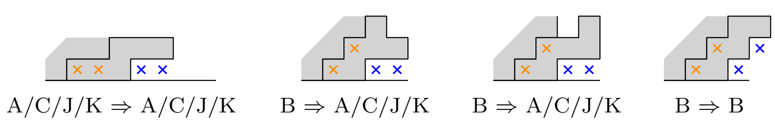

Fig. 10. The four cases of covering both problematic cells (with any number of nets). The blue X's denote new problematic cells created by the new nets.

Notice that all four development placements create two new problematic cells either in the arrangement A/C/J/K or B. Thus covering both problematic cells "propagates" to a new pair of problematic cells in one of the same two arrangements A/C/J/K or B. These problematic cell propagations follow one of the rectangle sides incident to the corner where the first pair of problematic cells were placed.

If such a propagation continues, it eventually reaches another corner of the rectangle (Fig. 11) or a pair of problematic cells propagated from an adjacent

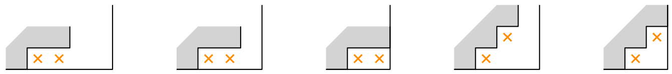

Fig. 11. The cases of a problematic cell arrangement near a corner. In each case, at least two cells are left uncovered.

corner (Fig. 12). In both cases, at least one of the problematic cells is left uncovered.

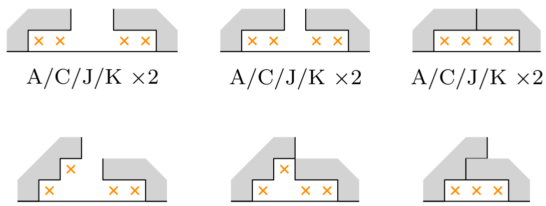

Fig. 12. The cases of a problematic cell arrangement meeting another problematic cell arrangement (each propagated from a different corner). In each case, at least one cell of each arrangement is left uncovered.

In summary, the key observations of the proof are:

- A rectangle has "four" corners.
- Each corner has an uncovered cell or problematic cell arrangement that contains two cells.
- Covering both cells of a problematic cell arrangement creates ("propagates") a new problematic cell arrangement.
- If a problematic cell arrangement is adjacent to a corner, at least one cell is left uncovered.
- If a problematic cell arrangement is adjacent to another problematic cell arrangement (originating at another corner), at least two cells are left uncovered.

Thus at least four uncovered cells are always found in any packing of nets in the rectangle.

Notice that the above argument assumed for each corner that the initial net placed to cover the corner cell created a problematic cell arrangement *attributed to this corner*. If a single development covers multiple corners simultaneously, then this is no longer the case and the proof does not hold. Thus we assume the larger rectangle dimension (m in the lemma statement) is at least 6, implying that no net can cover non-adjacent corner cells simultaneously. It is easily seen that the only net that is able to cover adjacent corner cells is E (when $n = 3$), in which case two distinct uncoverable cells, each adjacent to one of the corner cells covered by E, are created. □

We now consider the remaining case.

Lemma 6. *For rectangles of dimensions $m \times n$ with $5 \geq m \geq n \geq 2$, any packing of nets of the cube leaves at least 4 uncovered cells.*

Proof. This is again verified by using BurrTools, seen in Table 1 (left). □

The previous two lemmas together imply the following result for all rectangles, excluding those with area less than 4, that any packing of nets of the cube into a rectangle leaves at least 4 uncovered cells, which is a tight bound.

4 Dicube Nets

Now we turn our focus to nets of a *dicube*, that is, (the surface of) a face-to-face gluing of two unit cubes. There is exactly one dicube (up to symmetry), and we can show by enumeration that it has 723 different nets.

We ask a similar primitive question to the case of the cube: can any combination of dicube nets exactly pack some rectangle? Surprisingly, in contrast to the cube case, the answer is affirmative.

Theorem 4. *Nets of the dicube can exactly pack a 26×20 rectangle.*

Proof. The proof is by demonstration, as we can see in Fig. 13. In total, the packing uses $26 \cdot 20/10 = 52$ nets, which come from 11 distinct nets (up to symmetry). □

This example was found by hand, so we do not know whether it is minimal.

Extending to tricubes (or other n-cubes) is not so simple because there are multiple tricube shapes.

5 Conclusion

We conclude the paper by posing some open questions and conjectures.

As we can observe in Table 1, the number of uncovered cells in all the cases is not greater than 12. We strongly believe that it is upper bounded by 12, and that our upper bound 14 s not tight. Therefore the first open question is the following.

Question 1. Can we prove that the upper bound on the number of uncovered cells is 12?

Furthermore, we believe the following, which partially supports Odawara's conjecture [15].

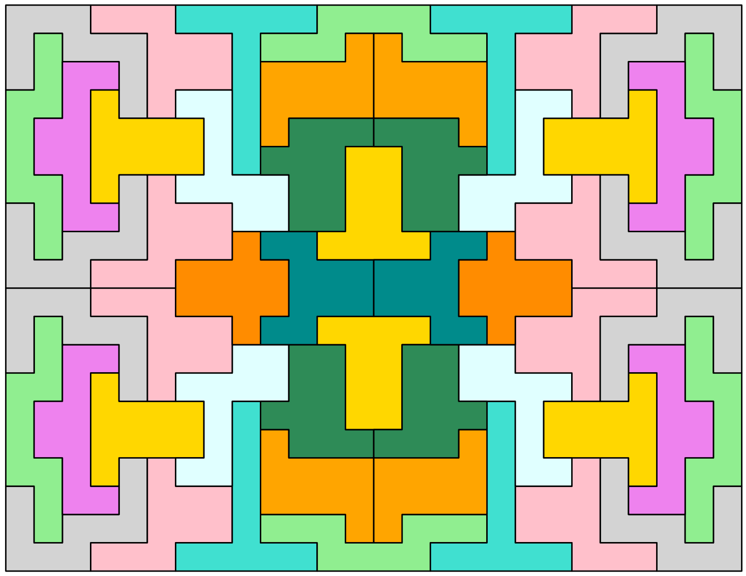

Fig. 13. A rectangle exactly packed by developments of the dicube.

Conjecture 1. For a $6 \times 6k$ rectangle, the minimum number of uncovered cells is exactly 12 (exactly $6k-2$ nets are packed).

For nets of the dicube, as we stated in Section 4, the following question remains open.

Question 2. What is a rectangle of minimum area that can be exactly packed by nets of the dicube?

References

1. Zachary Abel, Brad Ballinger, Erik D. Demaine, Martin L. Demaine, Jeff Erickson, Adam Hesterberg, Hiro Ito, Irina Kostitsyna, Jayson Lynch, and Ryuhei Uehara. Unfolding and dissection of multiple cubes, tetrahedra, and doubly covered squares. *Journal of Information Processing*, 25:610–615, August 2017.
2. Maarten Bos. Tiling squares with two different hexominoes. *Cubism For Fun*, 70:4–7, July 2007.
3. Andrew L. Clarke. The poly pages: Polyominoes: Hexominoes. http://www.recmath.com/PolyPages/PolyPages/index.htm?Polyominoes.html#hexominoes, July 2005.
4. Karl Dahlke. Tiling rectangles with polyominoes. http://www.eklhad.net/polyomino/index.html, November 2017.
5. Karl A. Dahlke. The Y-hexomino has order 92. *Journal of Combinatorial Theory, Series A*, 51(1):125–126, 1989.
6. Erich Friedman. Polyominoes in rectangles. https://www2.stetson.edu/~efriedma/order/, April 2017.

7. Martin Gardner. Polyominoes. In *Hexaflexagons and Other Mathematical Diversions: The First Scientific American Book of Mathematical Puzzles and Games*, chapter 13, pages 124–140. University of Chicago Press, 1988. Originally published by Simon & Schuster, 1959, and since republished by MAA/Cambridge, 2008.
8. Martin Gardner. Polyominoes and rectification. In *Mathematical Magic Show*, chapter 13, pages 172–187 and 289–292. Mathematical Association of America, 1990. Originally published by Knopf, 1977.
9. S. W. Golomb. Checker boards and polyominoes. *The American Mathematical Monthly*, 61(10):675–682, 1954.
10. Solomon W. Golomb. Tiling with polyominoes. *Journal of Combinatorial Theory*, 1(2):280–296, 1966.
11. Solomon W. Golomb. *Polyominoes: Puzzles, Patterns, Problems, and Packings*. Princeton University Press, 2nd edition, 1994.
12. Tatsuya Inoha, Yoshitaka Inoue, Takayuki Ozawa, and Yushi Uno. Packing developments of cubes. In *Abstracts from the 20th Japan Conference on Discrete and Computational Geometry, Graphs, and Games*, 2017.
13. Tatsuya Inoha, Yoshitaka Inoue, Takayuki Ozawa, and Yushi Uno. Packing developments of cubes: a proof of the theorem. Manuscript, 2017.
14. David A. Klarner. Packing a rectangle with congruent N-ominoes. *Journal of Combinatorial Theory*, 7(2):107–115, 1969.
15. Mitsuhiro Odawara. Development of a cube (in Japanese). http://www.torito.jp/puzzles/hako047.shtml, September 2008.
16. Ed Pegg, Jr. The Eternity Puzzle. http://www.mathpuzzle.com/eternity.html, May 2002.
17. Gerard Putter. Gerard's universal polyomino solver. https://gp.home.xs4all.nl/PolyominoSolver/Polyomino.html, February 2012.
18. Michael Reid. Rectifiable polyomino page. http://www.cflmath.com/Polyomino/rectifiable.html, October 2005.
19. Andreas Röver. Burrtools. http://burrtools.sourceforge.net/, October 2018.
20. Alex Selby. Eternity page. http://www.archduke.org/eternity/, May 2006.
21. George Sicherman. Hexomino oddities. https://userpages.monmouth.com/~colonel/6ominodd/index.html, September 2017.