

COVERING FOLDED SHAPES*

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ABSTRACT. Can folding a piece of paper flat make it larger? We explore whether a shape S must be scaled to cover a flat-folded copy of itself. We consider both single folds and arbitrary folds (continuous piecewise isometries $S \rightarrow \mathbb{R}^2$). The underlying problem is motivated by computational origami, and is related to other covering and fixturing problems, such as Lebesgue’s universal cover problem and force closure grasps. In addition to considering special shapes (squares, equilateral triangles, polygons and disks), we give upper and lower bounds on scale factors for single folds of convex objects and arbitrary folds of simply connected objects.

1 Introduction

We explore how folds can make an origami model larger, in the sense that Joseph Wu’s one-fold stegosaurus¹ cannot be covered by a copy of the square from which it is folded. In more technical terms, we consider how to cover all possible folded versions of a given shape by a scaled copy of the shape itself, with the objective of keeping the scale factor as small as possible.

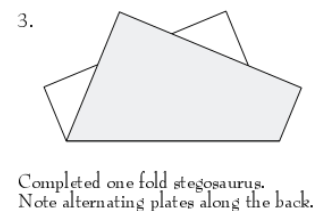


Figure 1: From Wu’s diagram.

*A preliminary version of this work was presented at the 25th Canadian Conference on Computational Geometry (CCCG’13) [1].

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¹An origami joke. <http://www.josephwu.com/Files/PDF/stegosaurus.pdf>

Problems of covering a family of shapes by one minimum-cost object have a long tradition in geometry. The classical prototype is Lebesgue’s universal cover problem from 1914 [11], which asks for a planar convex set of minimum area that can cover any planar set of diameter at most one; Brass and Sharifi [4] give upper and lower bounds, but a gap remains. A similar question, also with a gap, is Moser’s worm problem [10, 12], which asks for a convex set of minimum area that can cover any planar curve of unit length. As reported in the book by Brass, Moser, and Pach [3, Chapter 11.4], there is a large family of well-studied, but notoriously difficult problems parameterized by

- the family of sets to be covered,
- the sets allowed as covers,
- the size measure to be minimized, and
- the allowed transformations.

In this paper we consider a given *shape* S , which is a bounded region of the plane that is a simply connected (no holes) closed 2-manifold with boundary (every interior point has a disk neighborhood and every boundary point a half-disk). A shape S may possess more specific properties: e.g., it may be convex, a (convex or non-convex) polygon, a disk, a square, or an equilateral triangle.

We denote by cS , for $c > 0$, the family of copies of S that have been scaled by c , and then rotated, reflected, and translated. We consider upper and lower bounds on the smallest constant c such that, for any F obtained by folding S , some member of cS contains or *covers* F . Let us be more specific about folding.

A *single fold* of S with line ℓ reflects one or more connected components of the difference $S \setminus \ell$ across ℓ . Let $\mathcal{F}_1(S)$ denote the family of shapes that can be generated by a single fold of S . An *arbitrary fold* of S is a continuous, piecewise isometry from $S \rightarrow \mathbb{R}^2$, which partitions S into a finite number of polygons and maps each rigidly to the plane so that the images of shared boundary points agree. The key property of arbitrary folds is that the length of any path in S equals the length of its image in \mathbb{R}^2 . Let $\mathcal{F}(S)$ denote the family of shapes that can be generated by an arbitrary fold of S .

The single fold and arbitrary fold are two simple notions of flat folding that avoid concerns of layering and fold order. Note that any upper bound that we prove for arbitrary folds applies to single folds, too. And, although the image of an arbitrary fold need not be the result of single folds, our lower bounds happen to be limits of finite sequences of single folds. Our results apply to 3-d folded shapes if *covering* is understood to mean covering the orthogonal projection to the plane.

Throughout this paper, we consider the following type of covers:

Definition 1. For a given shape S and $c > 0$, cS is an *origami cover* of S if any member of $\mathcal{F}(S)$ can be covered by some member of cS . The *origami cover factor* $c^*(S)$ is the smallest such c , which may be ∞ :

$$c^*(S) = \inf\{c \mid cS \text{ is an origami cover of } S\}.$$

Analogously, cS is a 1-fold cover of S if any member of $\mathcal{F}_1(S)$ can be covered by some member of cS ; and the 1-fold cover factor $c_1^*(S)$ is the smallest such c :

$$c_1^*(S) = \inf\{c \mid cS \text{ is a 1-fold cover of } S\}.$$

Note that by definition $\mathcal{F}_1(S) \subseteq \mathcal{F}(S)$ and so $c_1^*(S) \leq c^*(S)$, for any shape S .

Questions of whether folding can increase area or perimeter have been considered before. It is clear that folding a piece of paper introduces overlap, so area can only decrease. On the other hand, the perimeter of a rectangle or square can be greater in a folded than an unfolded state—known as Arnold’s ruble note or the Margulis napkin problem [2, 8]. Folding techniques that increase perimeter, like rumpling and pleat-sinking, make very small but spiky models that are easily covered by the original paper shape, however.

Let us recall some common geometric parameters of shapes and derive a first simple general upper bound for the origami cover factor in terms of these parameters. For a given shape S , an *incircle*, C_r , is a circle of maximum radius (the *inradius* r) contained in S . Similarly, the *circumcircle*, C_R , is the circle of minimum radius (the *circumradius*) that contains S . In order to extend these notions to non-convex shapes, we consider geodesic distances, that is, the distance between two points in S is the length of a shortest path that connects the points and stays within S . The maximum geodesic distance D between any two points of S is the *geodesic diameter* of S . A *geodesic center* is a point in S that minimizes the maximum distance (the *geodesic radius* R) to all points of S . For convex shapes the geodesic radius R is also the circumradius. Jung’s theorem in the plane says $\sqrt{3}R \leq D \leq 2R$, with the equilateral triangle and circle giving the two extremes [13, ch. 16]. For any shape S , these parameters give an upper bound on the origami cover factor.

Lemma 2. *Any shape S with inradius r and geodesic radius R has an origami cover factor*

$$c^*(S) \leq R/r.$$

Proof. Place any folded state $F \in \mathcal{F}(S)$ in the plane so that the image of a geodesic center is at the origin. Choose a member of $(R/r)S$ with an incircle center at the origin. Because no path in F can be more than R from the origin, the scaled incircle covers F . \square

There are shapes for which the bound of Lemma 2 is tight. For instance, if S is a disk, then $r = R$ and $c^*(S) = 1$.

Results. The remainder of the paper is divided into two parts. First, in Section 2 we consider single folds and present bounds for the 1-fold cover factor of various families of shapes:

- If S is a convex shape with inradius r and circumradius R , then $c^*(S) \geq c_1^*(S) \geq \kappa R/r$, where $\kappa = ((\sqrt{5} - 1)/2)^{5/2} \approx 0.300283$. Note that this bound is within a constant factor of the general upper bound from Lemma 2.
- If S is an equilateral triangle, then $c_1^*(S) = 4/3$.

- If S is a square, then $c_1^*(S) = \varrho \approx 1.105224$, where ϱ denotes the largest (and only positive) real root of the polynomial $\Phi(x) = 40x^{12} + 508x^{11} + 1071x^{10} + 930x^9 - 265x^8 - 1464x^7 - 1450x^6 - 524x^5 + 58x^4 + 76x^3 + 3x^2 - 6x - 1$.
- If S is a polygon, then $c_1^*(S) > 1$, that is, for any polygon S there is a single fold such that the resulting folded state cannot be covered with a copy of S .
- On the other hand, we describe an infinite family of shapes for which $c_1^*(S) = 1$; these are shapes cut from a disk.

Then in Section 3 we discuss arbitrary folds and present a lower bound for the origami cover factor for a more general family of shapes:

- For a simply connected shape S with inradius r and geodesic radius R , we have $c^*(S) \geq \kappa R/r$, where $\kappa = \sqrt{3}/(2\pi) \approx 0.27566$.
- We describe a family of shapes that have an origami cover factor of 1, like disks. In fact, these shapes are constructed as a union of two disks.

2 Single Folds

In this section we explore the 1-fold cover factor $c_1^*(S)$, giving general bounds for convex S and for polygons, and the exact values for equilateral triangles, squares, and a family derived from disks.

2.1 Convex Shapes

For a convex set S , there is a lower bound for the 1-fold cover factor $c_1^*(S)$ that is within a constant factor of the upper bound given by Lemma 2.

Theorem 3. *Let S be a convex shape with inradius r and circumradius R . Then $\kappa R/r \leq c^*(S) \leq R/r$ for an appropriate constant $\kappa = ((\sqrt{5} - 1)/2)^{5/2} \approx 0.300283$.*

Proof. The upper bound is from Lemma 2.

For the lower bound, consider the center p^* of the circumcircle C_R that contains S . Because R is smallest possible, the set of points where the boundary of C_R touches S , $T := \partial C_R \cap S$, must contain at least two points, and no open halfplane through p^* can contain all of T . If $|T| = 2$, then these two points t_1 and t_2 must lie on a diameter of C_R ; if $|T| > 2$, there must be two points $t_1, t_2 \in T$ that form a central angle $\angle(t_1, p^*, t_2)$ in $[\frac{2}{3}\pi, \pi]$. Thus, for any $\varphi \in [0, \frac{2}{3}\pi]$, we can perform a single fold along a line through p^* that maps t_2 to t'_2 such that the central angle $\angle(t_1, p^*, t'_2)$ is φ .

Now, after folding, consider a cover of the three points t_1, p^*, t'_2 by cS for some $c > 0$. As each member of cS is convex, in covering the triangle $\Delta(t_1, p^*, t'_2)$, it also covers the largest circle C_Δ contained in $\Delta(t_1, p^*, t'_2)$; let r_φ be the radius of this circle, see Figure 2. Using elementary geometry we obtain $r_\varphi = \frac{R}{2} \frac{\sin(\varphi)}{1 + \sin(\varphi/2)}$, which is maximized at

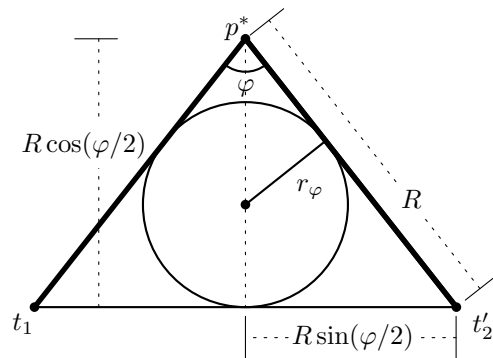
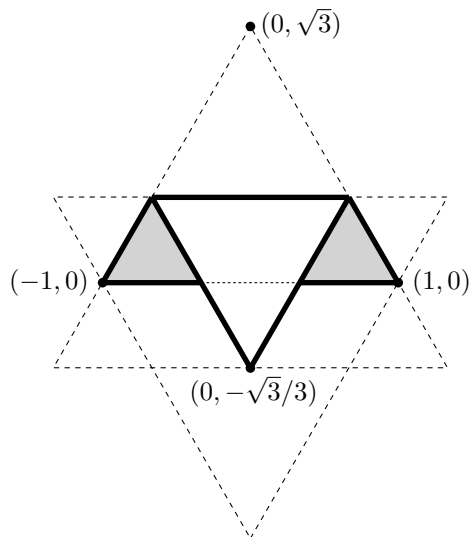


Figure 2: Parameters for calculating the 1-fold cover factor for convex S .

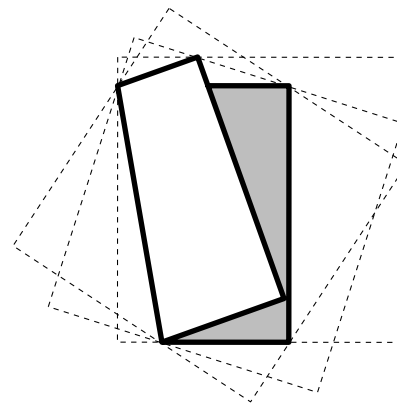
$\varphi = 2 \arctan(((\sqrt{5} - 1)/2)^{1/2}) \approx 76.345^\circ$, giving $r_\varphi = \kappa R$ as the radius of C_Δ . Because the largest circle covered by a member of cS has radius cr , and C_Δ is covered by cS , we conclude that $c \geq \kappa R/r$. \square

2.2 Cover Factors for Specific Polygons

In this section we determine $c_1^*(S)$ when S is an equilateral triangle or a square. These two cases illustrate analysis techniques that could in theory be extended to other polygons, except that the number of cases explodes, especially for non-convex shapes.



(a) Folded equilateral triangle and two minimum enclosing triangles with $c_1^* = 4/3$.



(b) Folded square and three minimum enclosing squares with $c_1^* \approx 1.105$.

Figure 3: Optimal 1-fold covers for equilateral triangle and square.

An important subproblem is to fix the folded shape F and compute, for the given shape S , the smallest c such that cS covers F . With four degrees of freedom for translation, rotation, and scaling, we expect that four first-order contacts between the boundaries of

S and F will define the minimum c . In polygons, these will be four pairs consisting of a vertex v of F and an edge e of S such that v lies on e .

For equilateral triangles, we can use the following reformulation of a lemma by DePano and Aggarwal:

Lemma 4 ([6, Lemma 2]). *The smallest enclosing equilateral triangle of a polygonal shape has at least one vertex of the shape on each side and at least one side of the triangle contains two vertices of the shape. (A vertex of the shape in the corner of the triangle counts for both incident sides.)*

When looking for an enclosing square, there is an additional possibility [5, 6]; the minimum may have four points in contact with four different sides. These papers compute such minima by solving for roots of polynomials, but an appealing direct construction of the square through four points, which is unique when it exists, is in Problem 20 in Kovanova and Radul’s list of “Jewish problems” [7]: for points A – D in ccw order, construct BD' perpendicular and of equal length to AC ; If $D' \neq D$, then two sides of the square must be parallel to DD' .

We use the following lemma, which can be found phrased slightly differently in Das et al. [5]:

Lemma 5 ([5]). *For any compact set $P \subset \mathbb{R}^2$ there exists a smallest enclosing square S of P that is of one of the following two combinatorial types:*

- (1) *each side of S contains a point from P ;*
- (2) *one side of S contains at least two points from P , and both the opposite side and an adjacent side of S each contain at least one point from P .*

These structural characterizations support the use of rotating calipers (see e.g. [14]) to compute minimum enclosing shapes. In what follows we show that the folds that define $c_1^*(S)$ are characterized by having multiple equal-sized enclosing shapes.

2.2.1 Equilateral Triangle

The example that establishes the maximum 1-fold cover factor of an equilateral triangle is nicely symmetric.

Theorem 6. *The 1-fold cover factor of an equilateral triangle, $c_1^*(\Delta)$, is $4/3$.*

Proof. Let S be the triangle of side length 2 with vertices $(\pm 1, 0)$ and $(0, \sqrt{3})$. We begin by showing that any single fold can be covered by scaling to at most $4/3$.

By symmetry, we may assume that we fold along a line $y = mx + b$ that intersects both edges incident on $(0, \sqrt{3})$; let P be the image of this vertex in the folded state $S' \in \mathcal{F}_1(S)$. For example, in Figure 3a, $P = (0, -\sqrt{3}/3)$. Consider three cases for the location of the image P and the resulting minimum enclosing equilateral triangle, depicted in Figure 4.

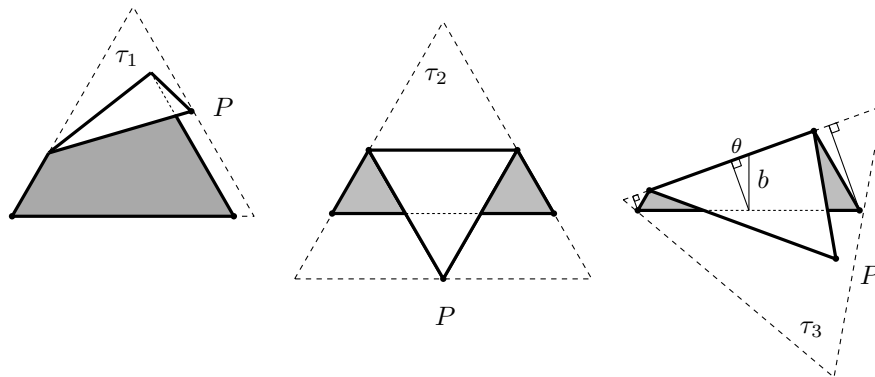


Figure 4: Cases for enclosing triangles depending on P . Point $P \in \tau_3$ should be below $P \in \tau_2$, but then the small triangles mentioned in the proof are hard to see.

First, suppose that P is on or above the x -axis. By symmetry, we may assume that P lies in the wedge formed by extending both edges of S incident on vertex $(-1, 0)$ to rays from $(-1, 0)$. Because P has distance at most 2 from $(-1, 0)$, scaling S about $(-1, 0)$ by $2/\sqrt{3} < 4/3$ creates an enclosing equilateral triangle τ_1 .

Second, suppose that the image $P = (p_x, p_y)$ has $-\sqrt{3}/3 \leq p_y \leq 0$. Consider the enclosing triangle τ_2 obtained by scaling S about $(0, \sqrt{3})$ until the horizontal edge touches P . The scale factor for this triangle is $\frac{\sqrt{3}-p_y}{\sqrt{3}} = 1 - p_y/\sqrt{3} \leq 4/3$.

Finally, suppose that $P = (p_x, p_y)$ has $p_y \leq -\sqrt{3}/3$. From the previous case, the scale factor for enclosing triangle τ_2 is $1 - p_y/\sqrt{3} \geq 4/3$. So instead consider an enclosing triangle τ_3 with an edge e along the fold line, which we can parameterize by its y -intercept $b \leq \sqrt{3}/3$ and angle from horizontal θ . Draw perpendiculars to e through vertices $(\pm 1, 0)$ to form two small 30-60-90 triangles. Edge e is composed of the short sides of these triangles plus the projection of the base edge of S , so e has length $(2 + 2b/\sqrt{3}) \cos \theta$. Thus, the scale factor of triangle τ_3 is $(1 + b/\sqrt{3}) \cos \theta \leq 4/3 \cos \theta \leq 4/3$.

These cases show that $c_1^*(\Delta) \leq 4/3$, and also reveal necessary conditions for equality: the fold line angle $\theta = 0$ and intercept $b = \sqrt{3}/3$, so $P = (0, -\sqrt{3}/3)$. To show that these are sufficient, we must check one more candidate for enclosing triangle.

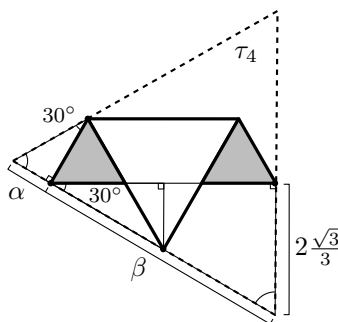


Figure 5: Not a minimum enclosing triangle.

Consider τ_4 , with edge incident to $P = (0, -\sqrt{3}/3)$ and $(-1, 0)$. The length of this edge is the sum of sides of two 30-60-90 triangles, marked α and β in Figure 5. The scale factor $(\alpha + \beta)/2 = \sqrt{3}/9 + 2\sqrt{3}/3 = 7\sqrt{3}/9 > 4/3$. Thus, τ_4 is *not* a minimum enclosing triangle, and $c_1^*(\Delta) = 4/3$, as determined by τ_2 and τ_3 . \square

2.2.2 Square

For squares, the fold that realizes the maximum 1-fold cover factor is astonishingly complex, and is neither symmetric, nor rational. For the unit square $[0, 1]^2$, the vertex $(0, 1)$ folds to a location whose y coordinate is the root of a degree twelve polynomial: $\Phi(x) = 40x^{12} + 508x^{11} + 1071x^{10} + 930x^9 - 265x^8 - 1464x^7 - 1450x^6 - 524x^5 + 58x^4 + 76x^3 + 3x^2 - 6x - 1$. This polynomial will arise because the optimal fold has *three* distinct minimum enclosing squares. Let ϱ denote the largest (and only positive) real root of $\Phi(x)$, which is approximately 1.105224.

Let $S = \{(x, y) : 0 \leq x, y \leq 1\}$ denote the axis-parallel unit square and consider some $F \in \mathcal{F}_1(S)$ such that $F \neq S$. Note that F is a simple polygon that is uniquely determined (up to symmetry) by a fold line ℓ .

Proposition 7. *The polygon F can be covered by S , unless fold line ℓ intersects S in the relative interior of two opposite sides.*

Proof. If ℓ does not intersect the interior of S then $F \cong S$. Otherwise ℓ intersects ∂S in exactly two points. If these points lie on adjacent sides of S , then folding along ℓ reflects the triangle formed by these sides and ℓ inside the portion of the square S on the opposite side of ℓ . Therefore, F can be covered by S . \square

We are interested in a fold line ℓ that maximizes the smallest enclosing square of F . Using symmetry with Proposition 7, we can assume:

- (1) the line ℓ intersects both horizontal sides of S (else rotate by 90°);
- (2) the slope of ℓ is negative (else reflect vertically);
- (3) ℓ intersects the top side of S left of the midpoint $(1/2, 1)$ (else rotate by 180°).

If we imagine F as the result of folding the part of S to the left of ℓ over to the right, then we can parameterize ℓ by the image $P = (p_x, p_y)$ of the top left corner $(0, 1)$ of S under this fold. Under the above assumptions, a line ℓ that passes (almost) through $(1/2, 1)$ and $(1, 0)$ would maximize p_y . Therefore $0 < p_x < 4/5$ and so $1 < p_y < \sqrt{2p_x - p_x^2} + 1 < 7/5$.

Denote the two points of intersection between ℓ and ∂S by $B = (b_x, 0)$ and $T = (t_x, 1)$ and denote the image of the bottom-left corner $(0, 0)$ of S under the fold across ℓ by $Q = (q_x, q_y)$. If $q_x > 1$, then the convex hull $\mathcal{CH}(F)$ of F is the hexagon $B, (1, 0), Q, (1, 1), P, T$, else Q does not appear on $\partial(\mathcal{CH}(F))$ and it is only a pentagon. Note that in any case the width of F in the y -direction is greater than one, whereas the width in the x -direction is less than one.

For a given $P = (p_x, p_y)$, we have

$$\begin{aligned} \ell &: y = -\frac{p_x}{p_y - 1}x + \frac{p_x^2 + p_y^2 - 1}{2(p_y - 1)}, \\ T &= \left(\frac{p_x^2 + (p_y - 1)^2}{2p_x}, 1 \right), \\ B &= \left(\frac{p_x^2 + p_y^2 - 1}{2p_x}, 0 \right), \text{ and} \\ Q &= \left(\frac{p_x(p_x^2 + p_y^2 - 1)}{p_x^2 + (p_y - 1)^2}, \frac{(p_x^2 + p_y^2 - 1)(p_y - 1)}{p_x^2 + (p_y - 1)^2} \right). \end{aligned}$$

What does a smallest enclosing square σ of F look like? For the upper bound on the cover factor we consider three enclosing squares (Figure 6).

σ_1 is the smallest axis-parallel enclosing square, which has points B and $(1, 0)$ on the bottom side, P on the top, T on the left, and no point on the right.

σ_2 has points P and $(1, 1)$ on one side, B on the opposite side, and T on a third side.

σ_3 has points B , $(1, 0)$, $(1, 1)$, and T appearing in this order, each on a different side of σ_3 .

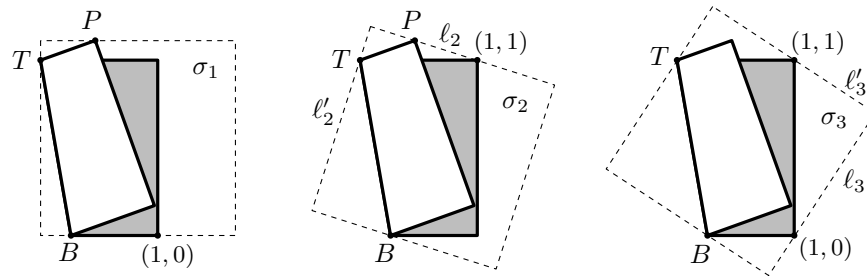


Figure 6: Three minimum enclosing squares for F .

Theorem 8. *The 1-fold cover factor of a square, $c_1^*(\square)$, is $\varrho \approx 1.105224$, where ϱ is the largest real root of the degree twelve polynomial $\Phi(x) = 40x^{12} + 508x^{11} + 1071x^{10} + 930x^9 - 265x^8 - 1464x^7 - 1450x^6 - 524x^5 + 58x^4 + 76x^3 + 3x^2 - 6x - 1$.*

Proof. The effort goes into showing that, for each folded shape F , one of the three enclosing squares σ_i , $i \in \{1, 2, 3\}$, as defined above, has side length at most ϱ .

Denote the side length of a square σ by $|\sigma|$. For a start it is easy to see that $|\sigma_1| = p_y < 7/5$, which provides a first upper bound.

For σ_2 we have to consider the distance $d(B, \ell_2)$, where ℓ_2 is the line through P and $(1, 1)$ and the distances $d((1, 0), \ell'_2)$ and $d(Q, \ell'_2)$, where ℓ'_2 is the line orthogonal to ℓ_2 through T . Noting that

$$d(B, \ell_2) = \frac{|p_x^2 p_y + p_y^3 + p_x^2 - 2p_x p_y - p_y^2 - p_y + 1|}{2p_x \sqrt{(p_x - 1)^2 + (p_y - 1)^2}}$$

$$d((1, 0), \ell'_2) = \frac{|p_y^2 p_x + p_x^3 - p_y^2 - 3p_x^2 + 2p_y + p_x - 1|}{2p_x \sqrt{(p_x - 1)^2 + (p_y - 1)^2}},$$

it can be checked that the former dominates the latter for $p_y \leq \frac{1}{2}(1 + \sqrt{4p_x - 4p_x^2 + 1})$ and that $d((1, 0), \ell'_2) > p_y$ for $\frac{1}{2}(1 + \sqrt{4p_x - 4p_x^2 + 1}) < p_y < \sqrt{2p_x - p_x^2 + 1}$ (and so $|\sigma_1| \leq |\sigma_2|$ in such a case). Exactly the same holds if $d((1, 0), \ell'_2)$ is replaced by

$$d(Q, \ell'_2) = \frac{|N_1|}{2p_x(1 + (p_x - p_y)^2)\sqrt{(p_x - 1)^2 + (p_y - 1)^2}},$$

where $N_1 = p_x^5 + 2p_x^3 p_y^2 + p_x p_y^4 - p_x^4 - 2p_x^3 p_y - 2p_x p_y^3 + p_y^4 - 4p_x^2 p_y - 4p_y^3 + 4p_x^2 + 2p_x p_y + 6p_y^2 - p_x - 4p_y + 1$. This verifies that σ_2 is enclosing, with side length $|\sigma_2| = d(B, \ell_2)$.

For σ_3 we consider a line $\ell_3 : y = m(x - 1)$ through $(1, 0)$, for some $m > 0$ and the orthogonal line $\ell'_3 : y = (m + 1 - x)/m$ through $(1, 1)$. If σ_3 is a smallest enclosing square, then $d(T, \ell_3) = d(B, \ell'_3)$. For our range of parameters, the only solution is

$$m = \frac{p_x^2 + p_y^2 - 1}{p_x^2 + (p_y - 1)^2},$$

which yields

$$|\sigma_3| = d(T, \ell_3) = \frac{\sqrt{2}|N_2|}{4p_x \sqrt{D_2}},$$

where $N_2 = p_x^4 + 2p_x^2 p_y^2 + p_y^4 - 4p_x^3 - 2p_x^2 p_y - 4p_x p_y^2 - 2p_y^3 + 4p_x p_y + 2p_y - 1$ and $D_2 = p_x^4 + 2p_x^2 p_y^2 + p_y^4 - 2p_x^2 p_y - 2p_y^3 + 2p_y^2 - 2p_y + 1$.

Because we choose the smallest square among σ_1 , σ_2 , and σ_3 , the claim certainly holds for $|\sigma_1| = p_y \leq \varrho$.

It can be checked that $|\sigma_2| \leq \varrho$, for all P with $\varrho < p_y < \sqrt{2p_x - p_x^2 + 1}$, except for a small region \mathcal{R} . This region \mathcal{R} is bounded from below by the line $y = \varrho$ and from above by the curve $\gamma : |\sigma_2| = \varrho$ (the branch of this curve that lies in $\{(x, y) : \varrho \leq y < \frac{1}{2}(1 + \sqrt{4x - 4x^2 + 1})\}$). The curve γ intersects the line $y = \varrho$ at two points, whose x -coordinates are approximately 0.67969 and 0.77126, respectively. The more interesting of these two is the first point of intersection, which can be described exactly as the smallest positive real root x_ϱ of the polynomial $40x^{12} - 116x^{11} - 1045x^{10} + 4756x^9 - 10244x^8 + 7260x^7 - 8392x^6 - 184x^5 + 620x^4 - 160x^3 + 1088x^2 - 192x + 256$. For the fold defined by $P = (x_\varrho, \varrho)$ we have $|\sigma_1| = |\sigma_2| = |\sigma_3| = \varrho$, while for all other points in \mathcal{R} the corresponding value for $|\sigma_3|$ is strictly less than ϱ .

It can also be checked that $|\sigma_3| < \varrho$, for any P with $p_y > \varrho$ and $\frac{1}{2}(1 + \sqrt{4p_x - 4p_x^2 + 1}) < p_y < \sqrt{2p_x - p_x^2 + 1}$ (above we committed to using σ_2 only if $p_y \leq \frac{1}{2}(1 + \sqrt{4p_x - 4p_x^2 + 1})$).

Altogether it follows that $\min\{|\sigma_i| : i \in \{1, 2, 3\}\} \leq \varrho \approx 1.105224446$, as claimed.

For the other direction, consider the fold defined by $P = (x_\varrho, \varrho)$, for which $|\sigma_1| = |\sigma_2| = |\sigma_3| = \varrho$. Using Lemma 5, it is easy to check that $\sigma_1, \sigma_2, \sigma_3$ are exactly the minimum enclosing squares for F . \square

2.3 Polygons

In contrast to disks, polygons can always be made larger with a single fold; that is, $c_1^*(P) > 1$ for all shapes bounded by a finite cyclic sequence of vertices and edges with no self-intersections.

Lemma 9. *For every plane polygon P , the 1-fold cover factor $c_1^*(P) > 1$.*

Proof. We look for finite sets of structures in P that, if not destroyed by folding, can only be covered by members of that set. For example, the set of diameters in a polygon is finite because the maximum distance D is realized only by pairs of vertices, and any diametral pair that remains at distance D in the folded state F must be covered by a diameter of P , possibly itself.

We proceed through increasingly elaborate structures as we consider different cases for the polygon P . To begin simply, suppose that in P there exist vertices that participate in two or more diametral pairs. (E.g., for odd n , every vertex of a regular n -gon.) Choose as our structure two diametral pairs, pq and qr , that minimize $\theta = \angle pqr$. Fold along a line trisecting θ , reflecting qr to create qr' in the folded shape F . This modified structure has angle $\angle pqr' = \theta/3$ between two diameters; by minimality of θ , it cannot be covered by P .

Normalize so that the diameter length $D = 1$. Choose $0 < \varepsilon < 1/4$ so all edge lengths in P are at least 2ε and the distance between any non-diametral pair of vertices is at most $1 - \varepsilon$. Define a *double arrow* to be a diameter pq , plus ε -length segments, ap , bp , cq and dq , of the polygon edges incident on p and q , respectively. Denote the positive angles these four edge form to pq by α , β , γ , and δ , respectively. Assume, by relabeling and reflecting, if necessary, that $\alpha + \beta \geq \gamma + \delta$ and $\gamma \geq \delta$, as illustrated in Figure 7. Note that angle $\angle paq$ is obtuse because a is at most the midpoint of its edge and the distance pq is at least the distance to the other endpoint. Similarly, the other three edges also form obtuse triangles with pq .

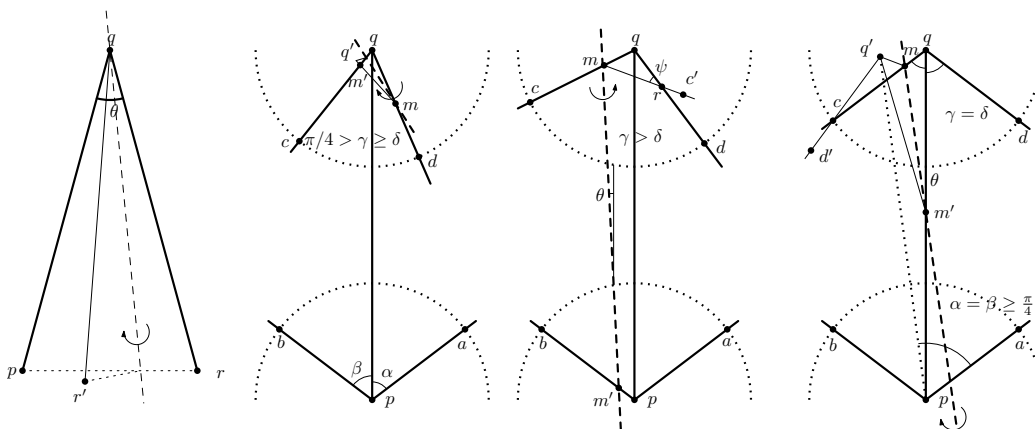


Figure 7: Structures in polygons and the folds used to show $c_1^*(P) > 1$.

From here on, we restrict attention to the set of *max arrows* that lexicographically maximize the 4-tuple $(\alpha + \beta, \gamma + \delta, \gamma, \max\{\alpha, \beta\})$. (That is, when comparing two 4-tuples,

compare the pair of j th elements if and only if for all $k < j$ the k th elements are equal.) Not only must max arrows be covered by max arrows, but we can find other structures within them.

Case $\gamma + \delta < \pi/2$: For this case we define a set of blunt arrow structures by removing the sharper tip of the max arrow. Let m be the midpoint of dq . Drop a perpendicular from m to cq at m' . A *blunt arrow* is any isometric copy of the heptagon $apbcm'md$. Because $mp > 1 - \varepsilon$, a blunt arrow can be covered only near a diameter, and because it includes portions of all four edges, these must be covered by a max arrow (possibly with reflections and rotations if enough of the angles have equal values.) Thus, a blunt arrow that is not destroyed by folding must be covered by one of a finite set of blunt arrows.

Folding the arrow tip q along the bisector of $\angle m'mq$ does not destroy the blunt arrow. However, by making mq' extend mm' , it does put q' outside any copy of P that covers this blunt arrow.

Case $\gamma + \delta \geq \pi/2$ and either $\gamma > \delta$ or $\alpha \neq \beta$: For this case we define a narrower dart structure. We have assumed that $\gamma > \delta$. Because we will use the fact that $\alpha + \beta \geq \gamma + \delta$ only to conclude that $\alpha + \beta \geq \pi/2$, the cases for $\alpha > \beta$ and $\beta > \alpha$ can be handled in the same way.

Let $q' = 2p - q$ be the point obtained by reflecting q at p . Draw a line through q' that forms an angle $\theta = (\gamma - \delta)\frac{\varepsilon}{6}$ with \overline{pq} , and intersects bp at m' and cq at m . (See the third subfigure of Figure ??.) A *dart* is a copy of the hexagon $apm'mqd$. Because a dart has a diameter segment and portions of all four incident edges, it should be clear that darts that survive folding to appear in F must be covered by darts in P .

The angle θ is chosen so that if we fold along the line $\overline{mm'}$, the reflected mc' intersects qd and extends outside the dart and the polygon. This claim can be proven as follows. For $0 < \theta < \pi/4$, we can bound $\frac{2\sqrt{2}\theta}{\pi} < \sin \theta < \theta$. Let $r = mc' \cap qd$ and apply the law of sines to $\triangle mrq$ to show that r is on both segments mc' and qd :

$$\begin{aligned} \angle mrq &= (\gamma - \delta) + 2\theta > (\gamma - \delta). \\ mq &= \frac{2 \sin \theta}{\sin(\gamma + \theta)} < 2\sqrt{2}\theta = \varepsilon \frac{\sqrt{2}(\gamma - \delta)}{3}. \end{aligned}$$

It follows that

$$qr < mr + mq = mq \left(\frac{\sin(\gamma + \delta)}{\sin(\angle mrq)} + 1 \right) < \varepsilon \frac{\sqrt{2}(\gamma - \delta)}{3} \left(\frac{\pi}{2\sqrt{2}(\gamma - \delta)} + 1 \right) < \varepsilon \left(\frac{\pi}{6} + \frac{\sqrt{2}\pi}{3 \cdot 4} \right) < \varepsilon.$$

Case $\alpha = \beta \geq \pi/4$ and $\gamma = \delta \geq \pi/4$: Because of $\alpha + \beta \geq \gamma + \delta$, we have $\alpha \geq \gamma$.

For this case we define a family of bent arrows, parametrized by angle θ , which is the angle the fold line ℓ_θ makes with pq . We show that for sufficiently small θ , a triangle with side on pa and vertex q' cannot be covered by a copy of P .

To complete the specification of the fold line ℓ_θ , choose its point of intersection $m' = \ell_\theta \cap \overline{pq}$ at distance $h = \varepsilon \cos \theta / \cos(\gamma - \theta)$ from q . This makes the folded image $q'd'$ pass through c , placing q' outside of the arrow. Folding in the other direction, we see that

points in the neighborhood of c' go outside of qd . Thus, neither P , nor the reflection of P can cover the folded shape F by aligning pq to a diameter. On the other hand, pq must remain in the neighborhood of a diameter, because $|pq'| \geq D - \varepsilon$. (Recall that diameter length $D = 1$.)

If $\varepsilon > \cos \gamma$, then we can choose $\theta > 0$ sufficiently small that $h > 1$ so folding misses the diameter pq and forms a dart, as in the previous case. So assume that the fold line ℓ_θ crosses the segment pq .

If we calculate the position of $q' = m' + (h, 2\theta)$, we observe that the derivative with respect to θ is perpendicular to pq :

$$\frac{d}{d\theta}q' = \frac{2\varepsilon}{\cos \gamma}(q - p)^\perp.$$

Let e be a portion of the polygon edge giving pa , truncated to length 2ε . (Thus, a is the midpoint of e .) We consider covering the triangle with side e and vertex q' . Initially, assume that we try to do so by rotating an arrow with maximal angle α . Let C be the circle that is tangent to pb at p and has e as a chord. Let $o \neq p$ be the other point intersection $C \cap pq$.

Now, fix edge e in the plane and rotate polygon P , keeping e covered by keeping the endpoints of e in contact with the wedge bpa of angle 2α . Point \hat{p} moves on an arc of circle C and \hat{q} moves so that the unit segment $\hat{p}\hat{q}$ always passes through o . We can represent this curve (a portion of a limaçon) in polar form about the origin o , with ψ being the angle from oq , and compute its derivative, which is not perpendicular to pq :

$$\hat{q} = \left(1 - 2\varepsilon \frac{\sin(\psi + \alpha)}{\sin(2\alpha)}, \psi\right)$$

$$\frac{d}{d\psi}\hat{q} = \frac{\varepsilon}{\cos \alpha}(q - p)^\perp - \frac{\varepsilon}{\sin \alpha}(q - p).$$

This means that we can choose a small $\theta > 0$ so that, even in the widest arrow of angle 2α , the edge e forces q' outside the curve traced by \hat{q} . For narrower arrows, q' is forced even further outside. Thus, P cannot cover F without scaling.

This completes the proof that $c_1^*(P) > 1$ for any polygon P . □

2.4 Disks and Limaçons

A circle sector, folded toward the center, never leaves its circle. This makes it easy to observe that $c_1^*(\bigcirc) = 1$.

We extend this observation to create a family of shapes, parameterized by an angle $\varphi \in (0, \pi/2)$, that has 1-fold cover factor $c_1^*(S_\varphi) = 1$, but origami cover factor $c^*(S_\varphi) > 1$. Figure 8 shows four examples. We remove a circle segment defined by a chord AB at distance $\cos \varphi$ from the disk center C . We then add back the area bounded by small loops of limaçons traced by reflections of A and B across lines at distance $\cos \varphi$ from C . For

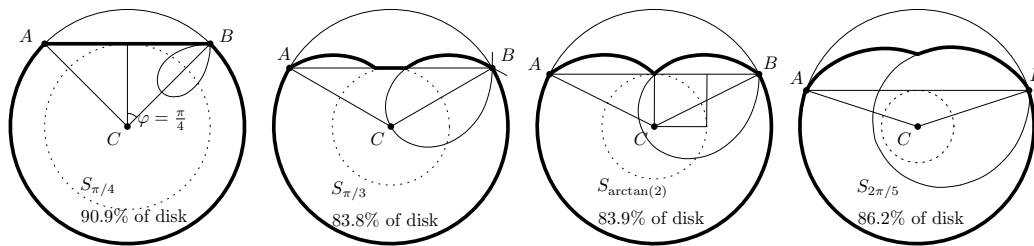


Figure 8: Example shapes S_φ with $c_1^*(S_\varphi) = 1$. The dotted circle of radius $\cos \varphi$ gives the drawn loops of limaçons in two ways: as reflections of B across tangents that separate B and C and as the envelope of circles through B that are centered on the arc of the dotted circle that lies between the tangents through B .

$\varphi \leq \pi/4$, the limaçons do not appear on the boundary of S_φ , and for $\varphi > \arctan(2) \approx 63.4^\circ$, the interior of AB does not appear.

First, we show that the reflection traces a limaçon, usually defined by tracing a point as one circle is rotated about another of the same radius. Its standard expression in polar coordinates is $r(\theta) = a + b \cos \theta$. Let C denote the circle of radius $\cos \varphi$ centered at C .

Lemma 10. *The loci of the reflections of A about all lines for which circle C is contained in one closed halfplane and point A in the other closed halfplane is the inner loop of a limaçon.*

Proof. For this proof only, choose a coordinate system with origin A , scaled so C is a unit circle with center at $C = (-d, 0)$ with $d > 1$. The boundary of the loci consists of the reflections of A about lines tangent to C , which can be parametrized by θ . Let $v = (\cos \theta, \sin \theta)$. The reflection of A about the line tangent to C at $C+v$ is $A - 2((A - C - v) \cdot v)v$, which can be written in polar form as $r(\theta) = 2 - 2d \cos \theta$. To see that we get the inner loop, notice that the tangents to the circle C through A are the extreme lines that satisfy the hypothesis, and these reflect A to itself. \square

With this formula, one can verify that the distance from C is unimodal, with the maximum at A and minimum at the other intersection of the loop with CA .

Another standard characterization of this loop of the limaçon² is as the intersection of disks centered on C and containing A . We know how disks fold, so:

Lemma 11. *Let ℓ be any line that does not intersect the arc of C that lies between the two tangents through A . The image of the limaçon loop by folding across ℓ is contained in the loop.*

Proof. Let ℓ^+ denote the closed halfplane of ℓ that contains the arc, and thus contains all centers of disks used to define the limaçon loop \mathcal{L} . For any individual disk O we fold a sector toward the center; the image of folding $O \setminus \ell^+$ across ℓ remains inside $O \cap \ell^+$. Since this is true for all disks, the portion of the loop $\mathcal{L} \setminus \ell^+$ folds inside the intersection $\mathcal{L} \cap \ell^+$. \square

²A demo by Daniel Joseph: <http://demonstrations.wolfram.com/LimaconsAsEnvelopesOfCircles/>

Now we can determine the 1-fold cover factor for S_φ .

Theorem 12. *The shape S_φ , with $\varphi \in (0, \pi/2)$, has 1-fold cover factor $c_1^*(S_\varphi) = 1$.*

Proof. For a given line ℓ , let ℓ^+ be the closed halfplane containing C . Construct a folded shape F by reflecting across line ℓ one or more components of $S_\varphi \setminus \ell^+$. We consider cases for the fold line ℓ based on its distance from C and the types of boundary curves of S_φ that it intersects.

Suppose first that the fold line ℓ is less than $\cos \varphi$ from C (i.e., intersects a dotted circle in Figure 8) and that all components of $S_\varphi \setminus \ell^+$ are folded over. Fold the entire unit disk along ℓ then rotate about C to make ℓ parallel to AB . Since this folded unit disk is covered by S_φ , the subset F is certainly covered by S_φ .

We sketch the arguments that, for all remaining cases, F remains inside S_φ with no rotation needed. It will be enough to consider points of $S_\varphi \setminus \ell^+$ that fold to locations above AB . This can only occur for $\varphi > \pi/3$, which happens to place C inside the two limaçon loops. (This is not crucial to the argument, but does simplify its geometric interpretation.)

We begin with the cases in which the fold line ℓ is at least $\cos \varphi$ from C . Lemma 11 implies that points in limaçon loops remain inside their loops after folding. So it is enough to consider any point Q on the arc of the unit circle that bounds S_φ between A and B . By Lemma 10, the image of Q under all possible folds in this case is also a limaçon loop; we can rotate this loop around C to align with the loop at A or B .

For our chosen fold line ℓ we get an image Q' above AB . If we extend CQ' through Q' we hit the loop for Q before we hit the last of the loops for A or B , thanks to unimodality of distance to C . Thus Q' is inside S_φ .

In the remaining case $S \setminus \ell^+$ has two components and we choose to fold only one. One component must be a portion of a limaçon loop that ℓ intersects twice above AB ; Lemma 11 implies that these may be safely folded. So we may assume that the fold line ℓ has positive slope, intersects the loop of B twice above AB , and intersects the dotted circle C in the upper left quadrant. If ℓ crosses the boundary of S_φ on AB , then all points of the folded component go below AB . Otherwise, ℓ crosses on the limaçon loop for A . The centers of disks that define the portion of this loop above AB are in the lower left quadrant of C , so lie in ℓ^+ . As in the proof of Lemma 11, points that fold above AB remain in the intersection of these folded disks, and thus remain in the loop. \square

To see that the origami cover factor $c^*(S_\varphi) > 1$, find the two points of S_φ farthest from $(0, -1)$, then crimp fold to narrow the angle between them.

The minimum area of S_φ is achieved by $\varphi \approx 61.5^\circ$, which occupies about 83.7% of the unit disk. The shape is between $S_{\pi/3}$ and $S_{\arctan(2)}$ in Figure 8.

3 Arbitrary Folds

3.1 Simply-Connected Shapes

In this section we show that, for a simply connected shape S , there is a lower bound for the origami cover factor $c^*(S)$ that is within a constant factor of the upper bound given by Lemma 2.

Theorem 13. *Let S be a simply connected shape with inradius r , geodesic radius R , and geodesic diameter D . Then $\kappa R/r \leq D/(2\pi r) \leq c^*(S) \leq R/r$ for $\kappa = \sqrt{3}/(2\pi) \approx 0.27566$.*

Proof. Again, the upper bound is from Lemma 2. The basic idea for the lower bound is to find a path in S that can be folded into a large circle, which must then be covered by a scaled copy of the incircle of S . Here, for brevity, we use a path of length D , the geodesic diameter.

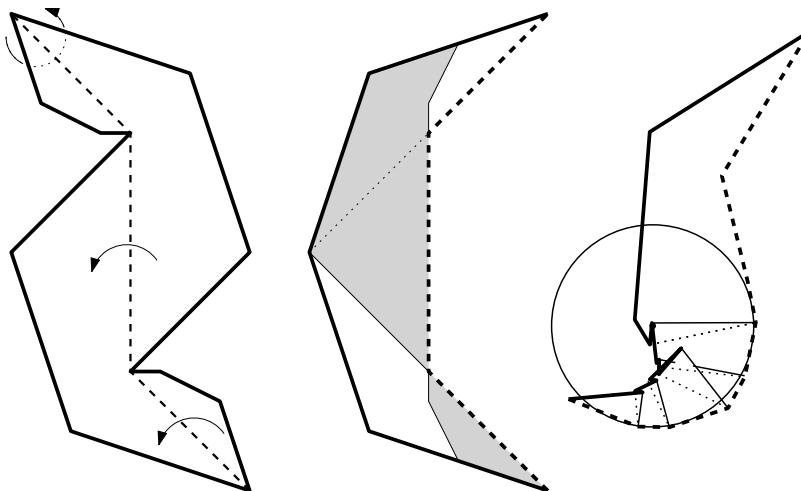


Figure 9: For Theorem 13, folding inflection edges to make a generalized spiral, then crimping to approximate a circle that must be covered by the incircle.

A *generalized spiral* is a simply connected region composed of consistently orientable plane patches having a distinguished shortest path γ that follows the boundary and never turns to the left. A generalized spiral may overlap itself if projected onto a plane, but we can think of it as embedded in a covering space of the plane.

Ordinarily, a diameter path γ will alternate between sequences of left turns and right turns at boundary points; a portion of the path between opposite turns is a line segment that we can call an *inflection edge*. We can simply fold along every inflection edge, gluing doubled layers along these edges, to turn γ into a path that goes only straight or to the right. Folding any non-boundary edges creates a generalized spiral with path γ . These folds are along lines of the geodesic path, so γ remains a shortest path between its endpoints.

We fold the generalized spiral into a left-turning circle with circumference approaching the length of γ . If we sweep a paired point and normal vector along γ , we can think

of painting a portion of the generalized spiral with fibers that each start on γ and grow orthogonal to a local tangent (because γ is a shortest path) and that are disjoint (because the sweep in position and angle is monotonic). We construct a circle whose circumference is arbitrarily close to the length of γ by crimp folds that align successive fibers of γ with the circle center. Figure 9 shows an example. It does not matter how far the fibers extend towards or beyond the circle; in order to cover the boundary of the circle, the inradius r must be scaled to the circle radius, which is $D/(2\pi)$. \square

3.2 Disks with Bumps

Because the radius of a disk is simultaneously the inradius and the geodesic radius, Lemma 2 implies that the cover factor of a disk, $c^*(\bigcirc)$, is 1. It is interesting to note that there are other shapes S with $c^*(S) = 1$; here is one simple family.

In a unit disk centered at C with a chord AB , choose a point D between C and the midpoint of AB . Add the disk centered at D of radius $|AD|$. Thus, we have a family of shapes $S_{d,e}$, parameterized by two distances, $d = |CD|$ and $e =$ distance from C to chord AB , satisfying $0 < d \leq e < 1$. See Figure 10.

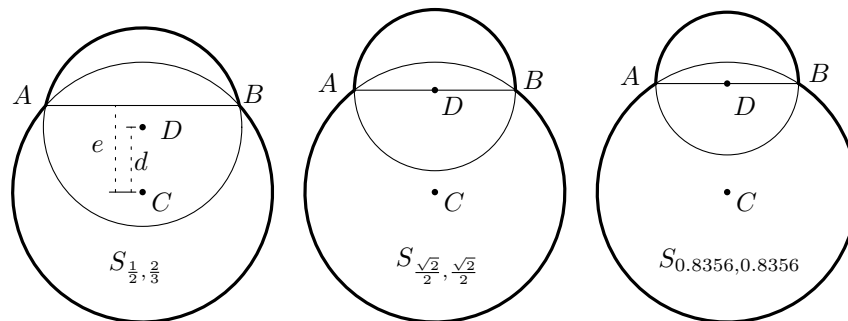


Figure 10: Shapes $S_{d,e}$ with $c^*(S_{d,e}) = 1$.

Lemma 14. *The shape $S_{d,e}$, with $0 < d \leq e < 1$, has origami cover factor $c^*(S_{d,e}) = 1$.*

Proof. Shape $S_{d,e}$ is the union of a unit disk centered at C and a disk centered at D whose radius we denote r . Note that by construction the boundaries of the disks intersect at A and B . This shape also covers all disks of radius r that are centered between C and D .

Now, in an arbitrary folded state $S'_{d,e}$, consider the locations of these centers, C' and D' . Placing a unit disk centered at C' and a radius r disk centered at D' will cover all points of $S'_{d,e}$. Because $|C'D'| \leq |CD|$, this pair of disks will be covered by placing a copy of $S_{d,e}$ with C at C' and D on the ray $\overrightarrow{C'D'}$. \square

Choose any $d \in (0, 1)$ and for all $e \in [d, 1)$ shape $S_{d,d}$ covers $S_{d,e}$, so these extremal members of the family have AB as the diameter of the smaller disk. Just for the sake of curiosity, the example with $d = e = \sqrt{2}/2$ minimizes the ratio of inradius to circumradius, $R/r = (1 + \sin \theta + \cos \theta)/2 \approx 0.8284$, and the example with $d \approx 0.8356$ minimizes the fraction of the circumcircle covered, $(\pi(1 + \sin^2 \theta) + \sin 2\theta - \theta)/(\pi R^2) \approx 0.7819$.

4 Open Problems

The most interesting questions are whether $c^*(\Delta) = c_1^*(\Delta)$ and $c^*(\square) = c_1^*(\square)$, and whether we can completely characterize those shapes with origami or 1-fold cover factor of unity.

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