A Complete Classification of Tile-makers

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Abstract

We extend the study of Akiyama's *tile-makers*: surfaces whose developments all tile the plane. First, we prove that the developments of Akiyama's tile-makers are the prototiles that tile the plane isohedrally with rotation and without reflection. Second, we give a simple characterization of all closed (boundaryless) tile-makers and give three new tile-makers not known to Akiyama. Finally, we prove that the developments of tile-makers are the isohedral prototiles.

1 Introduction

A *surface* is a two-dimensional manifold with a finite polyhedral metric, and may have non-spherical topology or boundary. Surfaces without boundary are *closed*. The *developments* of a surface are the planar shapes obtained by cutting the surface. A vertex of a surface has *(Gaussian) curvature* equal to 360° minus the angles of the incident faces, and non-vertex points have 0° curvature (also called *flat*). A polygon is a *prototile* provided congruent copies of the polygon can cover the plane without gaps or overlap.

Akiyama [1] initiated the study of *tile-makers*: surfaces whose developments are all prototiles. He provided a set of 5 classes of (possibly degenerate) polyhedra whose surfaces are tile-makers (see Figure 1). He also proved that these are all convex polyhedra that are tile-makers.

2 Tile-maker Completeness

The tile-makers of Akiyama have developments that are prototiles of *isohedral* plane tilings: tilings where there exists a rigid mapping of the plane from any tile to any other tile that leaves the entire tiling invariant. Ignoring prototile symmetries, there are 9 types of isohedral tilings, and each uses tiles generated by distinct set of rotations and reflections of the prototile (see [2, 4]). We prove that the developments of Akiyama's tile-makers are all prototiles that admit isohedral tiling using at least two rotations and no reflections of the prototile. Thus we say that Akiyama's tile-makers are *complete* for these prototiles:

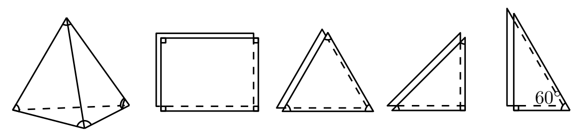

Figure 1: The tile-makers of Akiyama. From left to right: almost-regular tetrahedra, doubly covered rectangles, equilateral triangles, isosceles right triangles, and half equilateral triangles. The right four are degenerate polyhedra with two congruent faces.

Theorem 2.1. *Akiyama's tile-makers are complete for the prototiles that admit isohedral tilings using at least two rotations and no reflections of the prototile.*

3 Characterizing Tile-makers

The completeness of Akiyama's tile-makers for only a subset of the isohedral tiling types begs the question of whether there exist additional tile-makers complete for a larger class of prototiles. In the process of answering this question, we give a characterization of the closed surfaces that are tile-makers, also called *closed* tile-makers.

Lemma 3.1. *Let $\kappa(p)$ be the curvature of a point p. A closed surface S is a (closed) tile-maker if and only if for every $p \in S$, $\kappa(p) \geq 0$ and $360° - \kappa(p)$ divides $360°$.*

Characterizing a global property (tile-maker-ness) by a local property (curvature) is possible by using the well-known Gauss-Bonnet theorem. Indeed, our lemma can be thought of as a "Gauss-Bonnet theorem for tile-makers". The restriction to only closed

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tile-makers avoids "derivative" tile-makers obtained as partial cuttings of other tile-makers.

4 New Tile-makers

The Gauss-Bonnet theorem states that for any closed surface with Euler characteristic χ, the total curvature of the points on the surface is $360°\chi$. The Euler characteristic is closely related to the genus of a surface. For orientable and non-orientable surfaces with genus g, $\chi = 2 - 2g$ and $\chi = 2 - g$, respectively.

Applying the local curvature constraint of Lemma 3.1 to surfaces of Euler characteristic $0 \leq \chi \leq 2$ yields all closed tile-makers. The list consists of Akiyama's tile makers along with three new tile-makers, two of which are *flat*: they have $0°$ curvature at all points.

Theorem 4.1. *Flat tori, flat Klein bottles, and real projective planes flat everywhere except 2 points with* $180°$ *curvature are tile-makers. Along with Akiyama's, they are the only closed tile-makers.*

Klein bottles and real projective planes cannot be embedded in three dimensions without self-intersection. However, embeddable tile-makers can be obtained by partial cuttings of these surfaces, e.g. Möbius strips from Klein bottles. An example of obtaining a prototile as a development of a flat torus is seen in Figure 2.

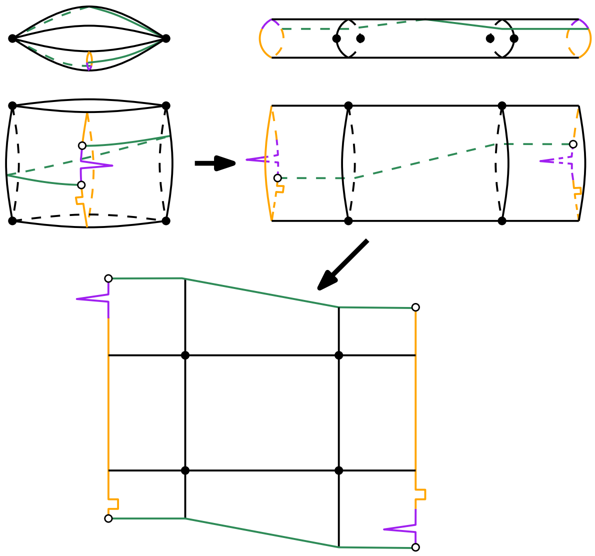

Figure 2: Unfolding a development of a flat torus into a prototile. Both side and top views are seen.

Perhaps as expected, Akiyama's tile-makers and the new tile-makers together are complete for the entire class of prototiles that admit isohedral tilings.

Theorem 4.2. *The set of all closed tile-makers (listed in Theorem 4.1) is complete for the set of prototiles that admit isohedral tilings.*

Not all prototiles admit isohedral tilings; such prototiles are called *anisohedral* (see Figure 3). By the previous result, anisohedral prototiles cannot be developments of closed tile-makers.

Corollary 4.3. *No anisohedral tile is the development of a closed tile-maker.*

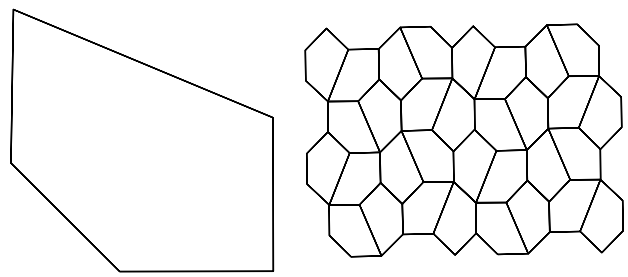

Figure 3: An anisohedral prototile due to Kershner [3]. Corollary 4.3 implies the prototile is not a development of any closed tile-maker.

References

[1] J. Akiyama. Tile-makers and semi-tile-makers. *The American Mathematical Monthly*, 114(7):602–609, 2007.

[2] B. Grünbaum and G. C. Shephard. *Tilings and Patterns*. W.H. Freeman and Company, 1987.

[3] R. B. Kershner. On paving the plane. *The American Mathematical Monthly*, 75(8):839–844, 1968.

[4] D. Schattschneider. *Visions of Symmetry: Notebooks, Periodic Drawings, and Related Work of M. C. Escher*. W. H. Freeman and Company, 1990.