Constrained Tri-Connected Planar Straight Line Graphs

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Abstract. It is known that for any set V of $n \geq 4$ points in the plane, not in convex position, there is a 3-connected planar straight line graph $G = (V, E)$ with at most $2n - 2$ edges, and this bound is best possible. We study the question whether this bound continues to hold if G is constrained to contain a given planar straight line graph $G_0 = (V, E_0)$. The answer in the affirmative if G_0 is a Hamiltonian cycle, or a matching such that none of the edges is a chord of the convex hull of V. However, the bound does not hold for some 3-regular graphs G_0.

1 Introduction

It is easy to see that for every $n \geq 4$, there is a 3-connected planar graph with n vertices and $\lceil 3n/2 \rceil$ edges (where all but at most one vertex has degree 3). On a set V of $n \geq 4$ points in the plane, however, a 3-connected planar straight line graph (PSLG) may require many more edges. García *et al.* [4] proved that if $3 \leq h < n$ points lie on the convex hull of V, then there is a 3-connected PSLG $G = (V, E)$ with at most $\max(\lceil 3n/2 \rceil, n+h-1) \leq 2n-2$ edges, and this bound is best possible. If the points are in convex position, then no PSLG (including a triangulation) is 3-connected.

Problem definition. We study how the minimum size of a 3-connected PSLG on a given point set is affected if E must contain a set of *constrained* edges. A PSLG $G_0 = (V, E_0)$ can be augmented to 3-connected PSLG $G = (V, E)$ if and only if V is not in convex position and E_0 does not contain any chord of the convex hull of V [6]. Such graphs are called *3-augmentable*. We pose the following questions:

(1) Under which edge constraints can the upper bound of $|E| \leq 2n - 2$ be maintained? That is, which 3-augmentable PSLGs $G_0 = (V, E_0)$ can be augmented to a PSLG $G = (V, E)$ with $|E| \leq 2n - 2$ edges?

(2) More generally, for a 3-augmentable PSLG $G_0 = (V, E_0)$ with $n \geq 4$ vertices, let $f(G_0)$ be the minimum size of an edge set E_1 such that (V, E_1) is a 3-connected PSLG, and let $g(G_0)$ be the minimum size of an edge set E_2 such that (V, E_2) is a 3-connected PSLG and $E_0 \subseteq E_2$. It is clear that $f(G_0) \leq g(G_0)$. For which graphs G_0 is $f(G_0) = g(G_0)$ possible? What is the behavior of the difference $g(G_0) - f(G_0)$?

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Results. In this note, we give partial answers to the first question. We show that if G_0 is a Hamiltonian cycle, not all vertices in convex position, then it can be augmented to a 3-connected PSLG with at most $2n - 2$ edges. Similarly, if G_0 is a 3-augmentable matching, then it can be augmented to a 3-connected PSLG with at most $2n - 2$ edges.

Al-Jubeh *et al.* [2] showed recently that this also holds if G_0 is the disjoint union of convex polygons lying in a triangle. We conjecture that the same holds for any 3-augmentable PSLG of maximal degree 2. However, there are 3-augmentable 3-regular PSLGs with $n \geq 4$ vertices for which any 3-connected augmentation has at least $\lfloor 9n/4 \rfloor - 1$ edges. Consider, for instance, the disjoint union of a chain of $n/4$ nested copies of K_4.

Related previous results. The analogous bounds for connectivity are straightforward. Every connected graph on n vertices has at least $n - 1$ edges, and every PSLG with maximum degree 0 or 1 can be augmented to a connected PSLG with $n-1$ edges. For bi-connectivity, however, similarly strong results are not possible. For every set V of $n \geq 3$ noncollinear points, there is bi-connected PSLG $G = (V, E)$ with at most n edges. But for some perfect matchings, every augmentation to a bi-connected PSLG has at least $\frac{3}{2}n - 2$ edges.

Al-Jubeh *et al.* [3] proved that every 3-edge-augmentable PSLG with n vertices can be augmented to a 3-edge-connected PSLG with at most $2n - 2$ new edges. However, this bound does not apply to vertex-connectivity, and the bound $2n - 2$ applies for the new edges, rather than the total number of edges.

2 Augmenting a simple polygon

Let H be a simple polygon in the plane with n vertices, that is, a straight line embedding of a Hamiltonian cycle. We show that it can be augmented to a 3-connected PSLG with at most $2n - 2$ edges. We use the following concept in our argument. For an (abstract) graph $G = (V, E)$, a subset $U \subseteq V$ is called *3-connected* if between any two vertices of U there are three disjoint paths in G. (Two paths between the same two vertices are called *disjoint* if they do not share any edges or vertices apart from their endpoints.) By Menger's theorem, if V is a 3-connected vertex set, then G is 3-connected. The following lemma holds for abstracts graphs.

Lemma 1 *Let $G = (V, E)$ be a graph such that $U \subset V$ is 3-connected subset of vertices. If G contains three disjoint paths from $v \in V \setminus U$ to three distinct vertices in U, then $U \cup \{v\}$ is also 3-connected.*

Proof. Suppose that G contains three disjoint paths from $v \in V \setminus U$ to distinct vertices $u_1, u_2, u_3 \in U$. It is enough to show that for every $u \in U$, there are three vertex disjoint paths between v and u. By Menger's theorem, it is enough to show that for any two vertices $w_1, w_2 \in V \setminus \{u, v\}$, the graph $G \setminus \{w_1, w_2\}$ contains a path between v and u. Since there are three disjoint paths between v and u_1, u_2, and u_3, graph $G \setminus \{w_1, w_2\}$ contains a path from v to u_i for some $i \in \{1, 2, 3\}$. If $u_i = u$, then we are done. Otherwise, $G \setminus \{w_1, w_2\}$ contains a path from u_i to u, since U is 3-connected. The union of these two paths (from v to u_i and from u_i to u) contains a paths from v to u. □

Lemma 2 *Let $H = (V, C)$ be a Hamiltonian cycle in a 3-connected (abstract) graph T with $n \geq 3$ vertices. Then T has a 3-connected subgraph G, $H \subseteq G \subseteq T$, that has at most $2n - 2$ edges.*

Proof. Let $uv \in T$ be an arbitrary chord of H. $H \setminus \{u, v\}$ is the disjoint union of two paths. Since $T \setminus \{u, v\}$ is connected, there is an edge $st \in T$ between some interior vertices of the two paths. Observe that $G_0 = H \cup \{uv, st\}$ is 3-connected, and so $U_0 = \{u, v, s, t\}$ is a 3-connected subset of vertices in G_0.

We augment G_0 with additional edges incrementally. Initially, let $i = 0$. While $U_i \neq V$, augment G_i with one new edge $e_i = s_i t_i$ to $G_{i+1} = G_i \cup \{s_i t_i\}$. We shall choose $s_i t_i$ such that at least one of its endpoints is not in U_i, and $U_{i-1} = \{s_i, t_i\}$ is 3-connected in G_{i+1}. By definition, each iteration increases the size of U_i and G_i by one. Thus, after at most $i \leq n - 4$ steps, we have $U_i = V$, and G_i is 3-connected.

We maintain the property that, for every $i \geq 0$, G_i can be represented as the union of interior disjoint paths between vertices of U_i, with at most one path between any two vertices of U_i. This clearly holds for G_0.

It remains the describe a general step i. Assume that $U_i \neq V$. Then there is a path P_i with at least one interior vertex. Denote the endpoints of P_i by $u_i, v_i \in U_i$. Since $T \setminus \{u_i, v_i\}$ is connected, there is an edge $e_i = s_i t_i \in T$ where s_i is an interior vertex of P_i and t_i is outside of P_i. Let $G_{i+1} = G_i \cup \{e_i\}$. We show that G_{i+1} contains disjoint paths from s_i to three distinct vertices of U_i. Path P_i contains two disjoint paths from s_i to the two endpoints of P_i. If $t_i \in U_i$, then $s_i t_i$ is the third path, and we can set $U_{i+1} = U_i \cup \{s_i\}$. Otherwise t_i is an interior vertex of another path P'_i, which has an endpoint in U_i different from the endpoints of P_i. Hence e_i and part of P'_i contains a path from s_i to a third vertex in U_i. Similarly, G_{i+1} contains disjoint paths from t_i to three distinct vertices of U_i. In both cases, set $U_i \cup \{s_i, t_i\}$ is 3-connected by Lemma 1. □

Corollary 3 *Every simple polygon on $n \geq 4$ vertices, not all in convex position, can be augmented to a 3-connected* PSLG *with at most $2n - 2$ edges.*

Proof. Let H be a nonconvex simple polygon on n vertices. By the results of Valtr and Tóth [6], H is 3-augmentable, and so there is a 3-connected PSLG T, in which H is a Hamiltonian cycle. Lemma 2 completes the proof. □

3 Augmenting disjoint line segments

Let M be a straight-line embedding of a matching with $n \geq 4$ vertices in general position in the plane. We show that if M is 3-edge-augmentable, then it can be augmented to a 3-connected PSLG which has at most $2n-2$ edges. We use the result by Hoffmann and Tóth [5] that M can be augmented to a Hamiltonian PSLG. It is not known, though, whether every straight line matching M can be augmented to a 2-regular PSLG (*c.f.* [1]).

Lemma 4 *Let T be a 3-connected Hamiltonian plane graph with $n \geq 4$ vertices, and let $M \subset T$ be a matching in T. Then T has a 3-connected subgraph G, $M \subseteq G \subseteq T$, with at most $2n - 2$ edges.*

In Lemma 4, we assume that T is a Hamiltonian plane graph. The lemma may hold even if exactly one of the Hamiltonicity or the planarity conditions is removed, but certainly not if both conditions are dropped: the complete bipartite graph $K_{3,n-3}$ is 3-connected for $n \geq 4$, it has $3n-9$ vertices, but it has no proper 3-connected subgraph.

Proof. Let H be an arbitrary Hamiltonian cycle in T. If $M \subset H$, then the result follows from Lemma 2. Suppose that $M \not\subset H$.

To construct a 3-connected graph G, $M \subseteq G \subseteq T$, incrementally, we will begin with a set $U_0 \subseteq V$ of four vertices which are 3-connected in a subgraph $G_0 \subseteq T$ with $n + 2$ edges. In step $i \geq 0$, we construct a subset $U_{i+1} \subseteq V$ of vertices and a subgraph $G_{i+1} \subseteq T$ such that U_{i+1} is a set of 3-connected vertices in G_{i+1}. We maintain that U_i is a proper subset of U_{i+1} and that G_{i+1} has at most $(n - 2) + |U_{i+1}|$ edges, but do **not** require that G_i is a subset of G_{i+1}. In other words, the set of edges in G_i may or may not increase monotonically. However, we require that G_i contains every edge of M induced by U_i. The algorithm terminates when $U_i = V$. At that time, G_i is a 3-connected spanning subgraph of T, and contains all edges of the matching M. The number of edges in G_i is at most $(n - 2) + |V| = 2n - 2$, as required.

Similarly to the proof of Lemma 2, we will also maintain the property that for every $i \geq 0$, G_i contains a set $\mathcal{P}_i$ of interior disjoint paths between distinct vertices in U_i. Between any two vertices of U_i, there is at most one path with some interior vertices (and at most one direct edge). Graph G_i is the union of the paths in $\mathcal{P}_i$, and possibly some edges of M between an interior vertex and an endpoint of the same path in $\mathcal{P}_i$.

Let P be a path in $\mathcal{P}_i$ or a proper subpath of some path in $\mathcal{P}_i$. We say that P is *dangerous* if (1) each endpoint u, v of P is connected to some interior point of P by an edge in $M \setminus G_i$, and (2) for every edge st in T between an interior vertex s of P and a vertex t outside of P, there is an edge in $M \setminus G_i$ between s and an endpoint of P (see Fig. 1). In our algorithm, we will maintain the invariant that $\mathcal{P}_i$ contains no dangerous path. To avoid dangerous paths, we also need the following definition. An interior vertex p of a path $P \in \mathcal{P}_i$ with endpoints u and v is *dangerous* if the subpath of P between u and p or between v and p is dangerous.

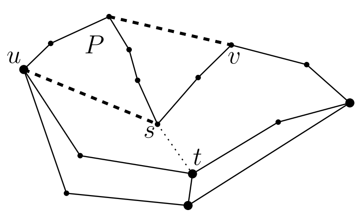

Figure 1: A dangerous path P between u and v, and a dangerous vertex v. Solid, dashed, and dotted edges are in G_i, $M \setminus G_i$, and $T \setminus (G_i \cup M)$, respectively.

Initialization. Let $uv \in M$ be an arbitrary chord of H. Now $H \setminus \{u, v\}$ is the disjoint union of two paths, each of which has some interior vertices. If an edge in M connects two interior vertices of the two paths, denote it by $st \in M$. Otherwise let st be an edge in $T \setminus M$ between two interior vertices of the two paths. In both cases, let $G_0 = H \cup \{uv, st\}$, in which $U_0 = \{u, v, s, t\}$ is a 3-connected subset of vertices. So U_0 has 4 vertices, G_0 has $n + 2$ edges, and the paths between vertices of U_0 are not dangerous.

Step *i*. Consider a general step $i \geq 0$, where we are given a 3-connected vertex set $U_i \subset V$, in a graph $G_i \subset T$, and a set of paths $\mathcal{P}_i$ between vertices in U_i, none of which is dangerous. We distinguish three cases. In all three cases, we augment G_i with an edge pq where p is an interior vertex of a path $P \in \mathcal{P}_i$, and add vertex p to U_i. If q happens to be an interior vertex of another path $P' \in \mathcal{P}_i$, then we add q to U_i as well, and we also augment G_i with any possible edge of $M \setminus G_i$ incident to q. This ensures that even if q is a dangerous vertex, the two subpaths of P' in $\mathcal{P}_{i+1}$ will not be dangerous. The three cases differ only on vertex $p \in P$.

Case 1. There is an edge pq in M such that p is an interior vertex of a path $P \in \mathcal{P}_i$ and q is outside of path P. (Fig. 2(a).) Let $U_{i+1} = U_i \cup \{p\}$ and $G_{i+1} = G_i \cup \{pq\}$. By Lemma 1, U_{i+1} is 3-connected in G_{i+1}, and p decomposes path P into two subpaths in $\mathcal{P}_{i+1}$, which are not dangerous.

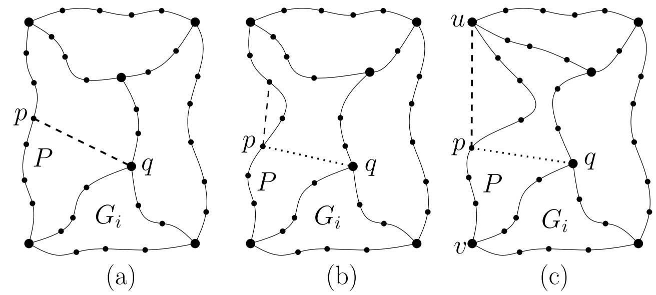

Figure 2: Cases 1-3a. Vertex p is in the interior of a path P and q is outside of path P. (a) Case 1: $pq \in M \setminus G_i$. (b) Case 2: $pq \in T \setminus M$ but there is no edge in $M \setminus G_i$ between p and an endpoint of P. (c) Case 3: $pq \in T \setminus M$ and there is an edge in $M \setminus G_i$ between p and an endpoint of P.

Case 2. Every edge in $M \setminus G_i$ connects vertices within the same path of $\mathcal{P}_i$. There is an edge pq in $T \setminus M$ such that p is an interior vertex of a path $P \in \mathcal{P}_i$, q is outside of path P, p is not dangerous, and there is no edge in $M \setminus G_i$ between p and an endpoint of path P. (See Fig. 2(b).) Let $U_{i+1} = U_i \cup \{p\}$ and $G_{i+1} = G_i \cup \{pq\}$. By Lemma 1, U_{i+1} is 3-connected in G_{i+1}, and p decomposes path P into two subpaths in $\mathcal{P}_{i+1}$, which are not dangerous.

Case 3. Every edge in $M \setminus G_i$ connects vertices within the same path in $\mathcal{P}_i$. For every edge $pq \in T$ between an interior vertex p of a path $P \in \mathcal{P}_i$ and a vertex q outside of that P, either p is dangerous or there is an edge in $M \setminus G_i$ between p and an endpoint of P. We consider two subcases.

Subcase 3a: There is an edge $pq \in T$ such that p is an interior vertex of a path $P \in \mathcal{P}_i$, vertex q is outside of P, and edge $m_p \in M \setminus G_i$ connects p to an endpoint of P. (Fig. 2(c).) Denote the two endpoints of P by u and v, and assume without loss of generality that $m_p = pu$. We would like to add p to U_i, but then we have to augment G_i with both pq and pu. We will augment U_i with three interior vertices of P. Vertex p decomposes path P into two paths: let $P_1 \subset P$ be the subpath between u and p, and P_2 between p and v. Note that P_1 has at least one interior vertex since $pu \in M \setminus G_i$, but P_2 may be a single edge. Since T is 3-connected, there is some edge st between an interior vertex s of P_1 and some vertex t outside of P_1. Observe that s cannot be a dangerous vertex, and there is no edge in $M \setminus G_i$ between s and an endpoint of P, otherwise P would be a dangerous path. Therefore t must be a vertex of P, i.e., either t is an interior vertex of P_2 or we have $t = v$. We examine both possibilities.

Subcase 3a(i): There is an edge st in T such that s is an interior vertex of P_1 and t is an in-

terior vertex of P_2. Let $U_{i+1} = U_i \cup \{p, s, t\}$ and $G_{i+1} = G_i \cup \{pq, pu, st\}$.

Subcase 3a(ii): For every edge st in T such that s is an interior vertex of P_1 and t is outside of P_1, we have $t = v$. We show that P_2 has no interior vertices. Suppose, to the contrary, that P_2 has interior vertices. Since T is 3-connected, there is an edge $s't'$ between an interior vertex s' of P_2 and a vertex t' outside of P_2. Note that there is no edge in $M \setminus G_i$ between s' and an endpoint of P, otherwise P would be dangerous, and s' is not dangerous, since $sv \in T$. Hence t' must be a vertex of path P. We have assumed that t' is not an interior vertex of P_1, and $t' \neq u$ because T is planar. Hence t' cannot be outside of P_2, which is a contradiction. We conclude that P_2 is a single edge $P_2 = \{pv\}$. Let $U_{i+1} = U_i \cup \{p, s\}$ and $G_{i+1} = (G_i \setminus \{pv\}) \cup \{pq, pu, sv\}$.

Subcase 3b. Every edge in $M \setminus G_i$ connects vertices within the same path in $\mathcal{P}_i$. For every edge $pq \in T$ between an interior vertex p of a path $P \in \mathcal{P}_i$ and a vertex q outside of that P, vertex p is dangerous. Denote the two endpoints of P by u and v. Vertex p decomposes path P into two paths: let $P_1 \subset P$ be the subpath between u and p, and P_2 between p and v. Assume without loss of generality that $P_1 \subset P$ is a dangerous path. Let p' and u' be the interior vertices of P_1 such that $pp', uu' \in M \setminus G_i$. Since T is 3-connected, T has an edge between an interior vertex of P_1 and a vertex outside of P_1. However, P_1 is a dangerous path, so only p' or u' may be connected to a vertex outside of P_1. Note that p' and u' are not dangerous vertices of P. Therefore, they can only be connected to some vertex in P. If there is an edge $p't$ between p' and an interior vertex of P_2, then let $U_{i+1} = U_i \cup \{p, p', t\}$ and $G_{i+1} = G_i \cup \{pq, pp', p't\}$. Similarly, if there is an edge $u't$ between u' and an interior vertex of P_2, then let $U_{i+1} = U_i \cup \{p, u', t\}$ and $G_{i+1} = G_i \cup \{pq, uu', u't\}$. Now assume that neither p' nor u' is adjacent to any interior vertex of P_2. Then at least one of them is adjacent to v. Similarly to case 3(a), we can show that P_2 is a single edge $P_2 = \{pv\}$.

Subcase 3b(i): The vertices u, p', u', p appear in this order along P_1. If $p'v \in T$, then let $U_{i+1} = U_i \cup \{p, p'\}$ and $G_{i+1} = (G_i \setminus \{pv\}) \cup \{pq, pp', p'v\}$. If $u'v \in T$, then let $U_{i+1} = U_i \cup \{p, u'\}$ and $G_{i+1} = (G_i \setminus \{pv\}) \cup \{pq, uu', u'v\}$.

Subcase 3b(ii): The vertices u, u', p', p appear in this order along P_1. Denote by P_3 the subpath of P between p and p'. Path P_3 has an interior vertex because $pp' \in M \setminus G_i$. Since T is 3-connected, there is an edge $s''t''$ in T such that s'' is an interior vertex of P_3, and t'' is outside of P_3. By our assumptions, t'' must be a vertex of P_1 (possibly $t'' = u$). Let $U_{i+1} = U_i \cup \{p, p', s'', t''\}$. If $t'' \in U_i$, then let $G_{i+1} = G_i \cup \{pq, pp', s''t''\}$; otherwise augment G_i with $\{pq, pp', s''t''\}$ and any edge in $M \setminus G_i$ incident to t''. $\square$

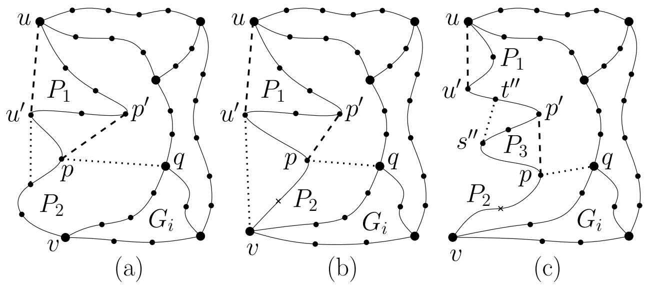

Figure 3: Cases 3b. Solid edges are part of graph G_i, dashed edges are in $M \setminus G_i$. p is a dangerous vertex in the interior of a path P. (a) Case 3b: There is an edge between u' and an interior vertex of P_2. (b) Case 3b(i): Vertices u, p', u', p appear in this order along P_1. (c) Case 3b(ii): Vertices u, u', p', p appear in this order along P_1.

Corollary 5 *Every 3-augmentable planar straight line matching with $n \geq 4$ vertices can be augmented to a 3-connected* PSLG *which has at most $2n - 2$ edges.*

Proof. Let M be a 3-augmentable planar straight line matching with $n \geq 4$ vertices. By the results of Hoffmann and Tóth [5], there is a PSLG Hamiltonian cycle H on the vertices of M that does not cross any edge in M. Since the Hamiltonian cycle H is crossing-free, none of its edges is a chord of the convex hull of vertices (otherwise the removal of this edge would disconnect H). Hence both H and $H \cup M$ are 3-augmentable [6]. That is, there is a 3-connected PSLG T on the same vertex set such that $M \cup H \subset T$. Lemma 4 completes the proof. $\square$

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