Spanning Properties of $\Theta\Theta_6$

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Let S be a set of n points in the plane. A simple undirected graph $G = (S, E)$ with vertex set S is called a *geometric t-spanner* if any two points $u, v \in S$ at distance $|uv|$ in the plane are at distance at most $t \cdot |uv|$ in G (the distance between two points in G is the length of a shortest path connecting them in G). The smallest integer t for which this property holds is called the *spanning ratio* of G.

For a fixed integer $k > 0$, the Yao graph $Y_k(S)$ and the Theta graph $\Theta_k(S)$ induced by S are constructed as follows. Partition the plane into k equiangular cones by extending k equally-separated rays starting at the origin, with the first ray in the direction of the positive x-axis, then translate the cones to each point $u \in S$, and connect u to a "nearest" neighbor in each cone. The difference between Yao and Theta graphs is in the way the "nearest" neighbor is defined. For a fixed point $u \in S$ and a cone $\mathcal{C}(u)$ with apex u, a Yao edge $\overrightarrow{uv} \in \mathcal{C}(u)$ minimizes the Euclidean distance $|uv|$ between u and v, whereas a Theta edge $\overrightarrow{uv} \in \mathcal{C}(u)$ minimizes the *projective distance* $\|uv\|$ from u to v, defined as Euclidean distance between u and the orthogonal projection of v on the bisector of $\mathcal{C}(u)$. Ties are arbitrarily broken. See Figure 1a.

Each of the graphs Θ_k and Y_k has out-degree at most k, but in-degree $n - 1$ in the worst case. To reduce the in-degrees, a second filtering step can be applied to the set of incoming edges in each cone. This filtering step eliminates, for each each point $u \in S$ and each cone with apex u, all but a "shortest" incoming edge. The result of this filtering step applied on Θ_k (Y_k) is the Theta-Theta (Yao-Yao) graph $\Theta\Theta_k$ (YY_k). Again, ties are arbitrarily broken.

Yao and Theta graphs (and their Yao-Yao and Theta-Theta sparse variants) have many important applications in wireless networking, motion planning and walkthrough animations. Many such applications take advantage of the spanning and sparsity properties of these graphs, which have been extensively studied. Molla [5] showed that, Y_2 and Y_3, are not spanners. On the other hand, it has been shown that, for any $k \geq 4$, Y_k and Θ_k are spanners (refer to [2]).

In contrast with Yao and Theta graphs, our knowledge about Yao-Yao and Theta-Theta graphs is more limited. Damian showed that, for $k \geq 5$, YY_{6k} and $\Theta\Theta_{6k}$ are 16.76-spanners [2] (and the asymptotic spanning ratio is $2+O(k^{-1})$). Recent breakthroughs show that YY_k, for all even $k \geq 42$, is a $(6.03 + O(k^{-1}))$-spanner [4], and YY_k for all odd $k \geq 3$ is not a spanner [3]. For small values $k \leq 5$, YY_k is not a spanner (refer to [2] and the references therein). Molla [5] showed that YY_6 is not a spanner, even for sets of points in convex position. This paper fills in one of the gaps in our knowledge of Theta-Theta graphs, proving that $\Theta\Theta_6$ is an 8-spanner for sets of points in convex position, but has unbounded spanning ratio for sets of points in non-convex position.

Our results. Our first result is based on an earlier result by Bonichon et al. [1] who showed that the *half-Θ_6-graph*, which is obtained by retaining only those edges of Θ_6 belonging to non-consecutive cones, is a plane 2-spanner. Here we establish the following result.

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Lemma 1 *For any edge $\overrightarrow{ab}$ in the Θ_6-graph induced by a set of points S in convex position, there is a path between a and b in $\Theta\Theta_6$ no longer than $4|ab|$, and this bound is tight.*

Figure 1b shows that the bound of Lemma 1 is tight. Combined with Bonichon's result, Lemma 1 establishes our first result that $\Theta\Theta_6$ is an 8- spanner for sets of points in convex position. This is the first result that marks a difference in the spanning properties of YY-graphs and $\Theta\Theta$-graphs.

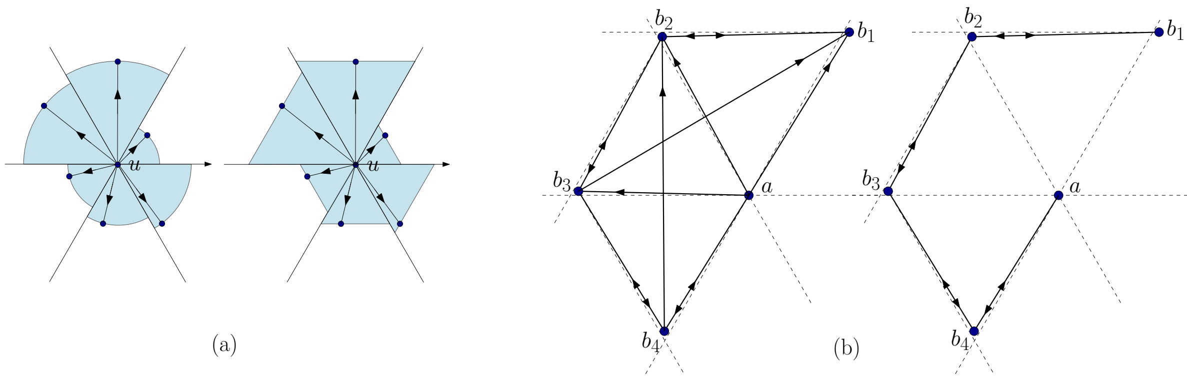

Figure 1: (a) Yao edges (left) minimize Euclidean distances and Theta edges (right) minimize projective distances (b) Θ_6-graph example (left) and corresponding $\Theta\Theta_6$-graph (right).

Our second result, depicted in Figure 2, shows that $\Theta\Theta_6$ is not a spanner for sets of points in non-convex position.

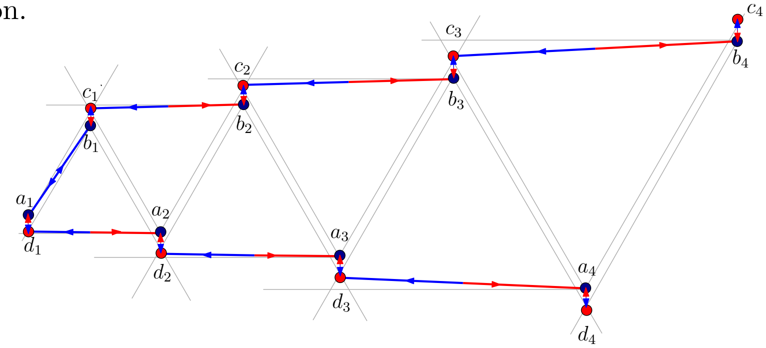

Figure 2: Point set $S = \{a_i, b_i, c_i, d_i \mid i = 1 \dots n\}$; $\Theta\Theta_6(S)$ is not a spanner.

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