Diffuse Reflection Radius in a Simple Polygon*

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Abstract. Light reflecting diffusely off of a surface leaves in all directions. It is shown that every simple polygon with n vertices can be illuminated from a single point light source s after at most $\lfloor (n-2)/4 \rfloor$ *diffuse reflections*, and this bound is the best possible. A point s with this property can be computed in $O(n \log n)$ time.

1 Introduction

When light diffusely reflects off of a surface, it scatters in all directions. This is in contrast to specular reflection, where the angle of incidence equals the angle of reflection. We are interested in the minimum number of diffuse reflections needed to illuminate all points in the interior of a simple polygon P with n vertices from a single light source s in the interior of P. A *diffuse reflection path* is a polygonal path γ contained in P such that every interior vertex of γ lies in the relative interior of some edge of P, and the relative interior of every edge of γ is in the interior of P (see Fig. 1 for an example). Our main result is the following.

Theorem 1. *For every simple polygon P with $n \geq 3$ vertices, there is a point $s \in \mathrm{int}(P)$ such that for all $t \in \mathrm{int}(P)$, there is an s-to-t diffuse reflection path with at most $\lfloor (n-2)/4 \rfloor$ internal vertices. This lower bound is the best possible. A point $s \in \mathrm{int}(P)$ with this property can be computed in $O(n \log n)$ time.*

Our main result is, in fact, a tight bound on the diffuse reflection radius (defined below) for simple polygons. Denote by $V_k(s) \subseteq P$ the part of the polygon illuminated by a light source s after at most k diffuse reflections. Formally, $V_k(s)$ is the set of points $t \in P$ such that there is a diffuse reflection path from s to t with at most k interior vertices. Hence $V_0(s)$ is the visibility polygon of point s within the polygon P. The *diffuse reflection depth* of a point $s \in \mathrm{int}(P)$ is the minimum $r \geq 0$ such that $\mathrm{int}(P) \subseteq V_r(s)$. The *diffuse reflection radius* $R(P)$ of a simple polygon P is the minimum diffuse reflection depth over all

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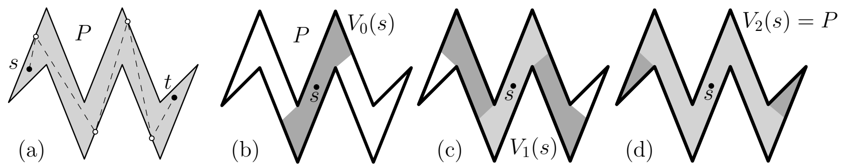

Fig. 1. (a) A diffuse reflection path between s to t in a simple polygon P. (b)–(d) The regions of a polygon illuminated by a light source s after 0, 1, and 2 diffuse reflections. The diffuse reflection radius of a zig-zag polygon with n vertices is $\lfloor (n-2)/4 \rfloor$.

points $s \in \text{int}(P)$. The set of points $s \in \text{int}(P)$ that attain this minimum is the *diffuse reflection center* of P. With this terminology, Theorem 1 implies that $R(P) \leq \lfloor (n-2)/4 \rfloor$ for every simple polygon P with $n \geq 3$ vertices. A family of zig-zag polygons (see such polygon in Fig. 1) shows that this bound is the best possible for all $n \geq 3$. The *diffuse reflection diameter* $D(P)$ of P is the *maximum* diffuse reflection depth over all $s \in \text{int}(P)$. Barequet et al. [6] recently proved, confirming a conjecture by Aanjaneya et al. [1], that $D(P) \leq \lfloor n/2 \rfloor - 1$ for all simple polygons with n vertices, and this bound is the best possible.

Proof Technique. The regions $V_k(s)$ are notoriously difficult to handle. Brahma et al. [7] constructed examples where $V_2(s)$ is not simply connected, and where $V_3(s)$ has $\Omega(n)$ holes. In general, the maximum complexity of $V_k(s)$ is known to be $\Omega(n^2)$ and $O(n^9)$ [2]. Rather than consider $V_k(s)$, we use the simply connected regions $R_k(s) \subseteq V_k(s)$ defined by Barequet et al. [6] (reviewed in Section 2.1) and prove that $\text{int}(P) \subseteq R_{\lfloor (n-2)/4 \rfloor}(s)$ for some point $s \in \text{int}(P)$.

In Section 2, we establish a simple sufficient condition (Lemma 1) for a point s to determine if $\text{int}(P) \subseteq R_{\lfloor (n-2)/4 \rfloor}(s)$. We use a generalization of the kernel of a simple polygon (Section 3.1) and the weak visibility polygon for a line segment (Section 3.2) to prove that there exists a point satisfying these considitions, with the exception of two extremal cases that are resolved directly (Section 2.2). The existential proof is turned into an efficient algorithm by computing the generalized kernel in $O(n \log n)$ time, and maintaining the visibility of a point moving along a line segment with a persistent data structure undergoing $O(n)$ updates in $O(\log n)$ time each.

Motivation and Related Work. The diffuse reflection path is a special case of a *link path*, which has been studied extensively due to its applications in motion planning, robotics, and curve compression [13,17]. The *link distance* between two points, s and t, in a simple polygon P is the minimum number of edges in a polygonal path between s and t that lies entirely in P. In a polygon P with n vertices, the link distance between two points can be computed in $O(n)$ time [20]. The *link diameter* of P, the maximum link distance between two points in P, can be computed in $O(n \log n)$ time [21]. The *link depth* of a point s is the smallest number d such that all other points in P are within link distance d of s. The *link radius* is the minimum over all link depths, and the *link center* is the set of points with minimum link depth. It is known that the link center is a convex region, and can be computed in $O(n \log n)$ time [12].

The *geodesic center* of a simple polygon is a point inside the polygon which minimizes the maximum internal (geodesic) distance to any point in the polygon. Pollack et al. [18] show how to compute the geodesic center of a simple polygon with n vertices in $O(n \log n)$ time. Hershberger and Suri [15] give an $O(n)$ time algorithm for computing the *geodesic diameter*. Bae et al. [5] show that the geodesic diameter and center under the L_1 metric can be computed in $O(n)$ time in every simple polygon with n vertices.

Note that the link distance, geodesic distance and the L_1-geodesic distance are all metrics, while the minimum number of reflections on a diffuse reflection path between two points is *not* a metric (the triangle inequality fails). This partly explains the difficulty of handling diffuse reflections.

In contrast to link paths, the currently known algorithm for computing a minimum diffuse reflection path (one with the minimum number of reflections) between two points in a simple polygon with n vertices takes $O(n^9)$ time [2,13]; and no polynomial time algorithm is known for computing the diffuse reflection diameter or radius of a polygon.

2 Preliminaries

For a planar set $U \subseteq \mathbb{R}^2$, we denote the interior by $\text{int}(U)$, the boundary by ∂U, and the closure by $\text{cl}(U)$. Let P be a simply connected closed polygonal domain (for short, *simple polygon*) with n vertices. A *chord* of P is a closed line segment ab such that $a, b \in \partial P$, and the relative interior of ab is in $\text{int}(P)$.

We assume that the vertices of P are in general position, and we only consider light sources $s \in \text{int}(P)$ that do not lie on any line spanned by two vertices of P. Recall that $V_0(s)$ is the visibility polygon of the point $s \in P$ with respect to P. The *pockets* of $V_0(s)$ are the connected components of $P \setminus \text{cl}(V_0(s))$. See Fig. 2(a) for examples. The common boundary of $V_0(s)$ and a pocket is a chord ab of P (called a *window*) such that a is a reflex vertex of P that lies in the relative interior of segment sb. We say that a pocket with a window ab is *induced by* the reflex vertex a. Note that every reflex vertex induces at most one pocket of $V_0(s)$. We define the *size* of a pocket as the number of vertices of P on the boundary of the pocket. Since the pockets of $V_0(s)$ are pairwise disjoint, the sum of the sizes of the pockets is at most n, the number of vertices of P.

A pocket is a *left* (resp., *right*) pocket if it lies on the left (resp., right) side of the directed line $\overrightarrow{ab}$. Two pockets of $V_0(s)$ are *dependent* if some chord of P crosses the window of both pockets; otherwise they are *independent*. One pocket is called independent if it is independent of all other pockets.

Proposition 1. *All left (resp., right) pockets of $V_0(s)$ are pairwise independent.*

The main result of this section is a sufficient condition (Lemma 1) for a point $s \in \text{int}(P)$ to fully illuminate $\text{int}(P)$ within $\lfloor (n-2)/4 \rfloor$ diffuse reflections. A proof of the lemma is offered in the full version of the paper. It relies on the following subsection, techniques developed in [6], and the bound $D(P) \leq \lfloor n/2 \rfloor - 1$ on the diffuse reflection diameter.

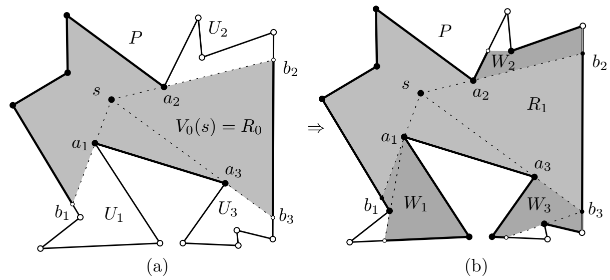

Fig. 2. (a) A polygon P where $V_0(s)$ has three pockets U_1, U_2 and U_3, of size 4, 3, and 5, respectively. The left pockets are U_1 and U_2, the only right pocket is U_3. Pocket U_1 is independent of both U_2 and U_3; but U_2 and U_3 are dependent. (b) The construction of region R_1 from $R_0 = V_0(s)$ in [6]. Pocket U_1 is saturated, and pockets U_2 and U_3 are unsaturated.

Lemma 1. *We have* $\text{int}(P) \subseteq V_{\lfloor (n-2)/4 \rfloor}(s)$ *for a point* $s \in \text{int}(P)$ *if the pockets of* $V_0(s)$ *satisfy these conditions:*

$\mathbf{C}_1$ *every pocket has size at most* $\lfloor n/2 \rfloor - 1$*; and*
$\mathbf{C}_2$ *the sum of the sizes of any two dependent pockets is at most* $\lfloor n/2 \rfloor - 1$.

2.1 Review of regions R_k.

We briefly review the necessary tools from [6]. Let $s \in \text{int}(P)$ be a point in general position. Recall that $V_k(s)$, the set of points reachable from s with at most k diffuse reflections, is not necessarily simply connected when $k \geq 1$ [7]. Instead of tackling $V_k(s)$ directly, Barequet et al. [6] recursively define simply connected subsets $R_k = R_k(s) \subseteq V_k(s)$ for all $k \in \mathbb{N}_0$, starting with $R_0 = V_0(s)$. We review how R_{k+1} is constructed from R_k. Each region R_k is bounded by chords of P and segments along the boundary ∂P. The connected components of $P \setminus \text{cl}(R_k)$ are the *pockets* of R_k. Each pocket U_{ab} of R_k is bounded by a chord ab such that a is a reflex vertex of P, b is an interior point of an edge of P, and the two edges of P incident to a are on the same side of the line ab (these properties are maintained in the recursive definition). A pocket U_{ab} of R_k is *saturated* if every chord of P that crosses ab has one endpoint in R_k and the other endpoint in U_{ab}. Otherwise, U_{ab} is *unsaturated.* Recall that for a point $s' \in P$, $V_0(s')$ is the set of points in P visible from s'; and for a line segment $pq \subseteq P$, $V_0(pq)$ is the set of points in P visible from any point in pq.

The regions R_k are defined as follows (refer to Fig. 2(b)). Let $R_0 = V_0(s)$. If $\text{int}(P) \subseteq R_k$, then let $R_{k+1} = \text{cl}(R_k) = P$. If $\text{int}(P) \not\subseteq R_k$, then R_k has at least one pocket. For each pocket U_{ab}, define a set $W_{ab} \subseteq U_{ab}$: If ab is saturated, then let $W_{ab} = V_0(ab) \cap U_{ab}$. If ab is unsaturated, then let $p_{ab} \in R_k \cap \partial P$ be a point infinitely close to b such that no line determined by two vertices of P separates b and p_{ab}; and then let $W_{ab} = V_0(p_{ab}) \cap U_{ab}$. Let R_{k+1} be the union of $\text{cl}(R_k)$ and the sets W_{ab} for all pockets U_{ab} of R_k. Barequet et al. [6] prove that $R_k \subseteq V_k(s)$ for all $k \in \mathbb{N}_0$.

We say that a region R_k *weakly covers* an edge of P if the boundary ∂R_k intersects the relative interior of that edge. On the boundary of every pocket U_{ab} of R_k, there is an edge of P that R_k does not weakly cover, namely, the edge of P incident to a. We call this edge the *lead edge* of U_{ab}. The following observation follows from the way the regions R_k are constructed in [6].

Proposition 2 ([6]). *For every pocket U of region R_k, $k \in \mathbb{N}_0$, the lead edge of U is weakly covered by region R_{k+1} and* is not *weakly covered by R_k.*

Proposition 3. *If a pocket U_{ab} of $V_0(s)$ has size m, then R_k weakly covers at least $\min(k+1, m)$ edges of P on the boundary of U.*

The following lemma is a direct consequence of Proposition 3. It will be used for unsaturated pockets of $V_0(s)$.

Lemma 2. *If U is a size-m pocket of $V_0(s)$, then $\mathrm{int}(U) \subseteq R_{m-1}$.*

For saturated pockets, the diameter bound allows a significantly better result.

Lemma 3. *If U is a size-m saturated pocket of R_k, then $\mathrm{int}(U) \subseteq R_{k+\lfloor m/2 \rfloor}$.*

Lemmas 2 and 3 combined yield the following for dependent pockets of $V_0(s)$.

Lemma 4. *Let U be a pocket of $V_0(s)$ of size m. If each pocket dependent on U has size at most $m' < m$, then $\mathrm{int}(U) \subseteq R_{\lfloor (m+m')/2 \rfloor}$.*

2.2 Double Violators

Recall that the sum of sizes of the pockets of $V_0(s)$ is at most n, the number of vertices of P. Therefore, it is possible that several pockets or dependent pairs of pockets violate conditions $\mathbf{C}_1$ or $\mathbf{C}_2$ in Lemma 1. We say that a point $s \in \mathrm{int}(P)$ is a *double violator* if $V_0(s)$ has either (i) two disjoint pairs of dependent pockets, each pair with total size at least $\lfloor n/2 \rfloor$, or (ii) a pair of dependent pockets of total size at least $\lfloor n/2 \rfloor$ and an independent pocket of size at least $\lfloor n/2 \rfloor$. (We do not worry about the possibility of two independent pockets, each of size at least $\lfloor n/2 \rfloor$.) In this section, we show that if there is a double violator $s \in \mathrm{int}(P)$, then there is a point $s' \in \mathrm{int}(P)$ (possibly $s' = s$) for which $\mathrm{int}(P) \subseteq V_{\lfloor (n-2)/4 \rfloor}(s')$, and such an s' can be found in $O(n)$ time.

The key technical tool is the following variant of Lemma 4 for a pair of dependent pockets that are adjacent to a common edge (i.e., *share* an edge).

Lemma 5. *Let U_{ab} and $U_{a'b'}$ be two dependent pockets of $V_0(s)$ such that neither is dependent on any other pocket, and points b and b' lie in the same edge of P. Let the size of U_{ab} be m and $U_{a'b'}$ be m'. Then $R_{\lfloor (m+m'-1)/2 \rfloor}$ contains the interior of both U_{ab} and $U_{a'b'}$.*

Lemma 6. *Suppose that $V_0(s)$ has two disjoint pairs of dependent pockets, each pair with total size $\lfloor n/2 \rfloor$. Then there is a point $s' \in \mathrm{int}(P)$ such that $\mathrm{int}(P) \subseteq V_{\lfloor (n-2)/4 \rfloor}(s')$, and s' can be computed in $O(n)$ time.*

Lemma 7. *Suppose that $V_0(s)$ has a pair of dependent pockets of total size $\lceil n/2 \rceil$ and an independent pocket of size $\lfloor n/2 \rfloor$. Then there is a point $s' \in \mathrm{int}(P)$ with $\mathrm{int}(P) \subseteq V_{\lfloor (n-2)/4 \rfloor}(s')$, and s' can be computed in $O(n)$ time.*

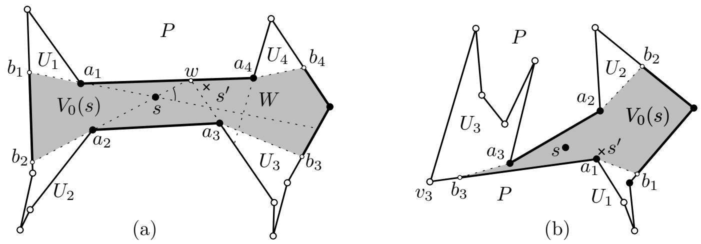

Fig. 3. Two instances of a double violator point s. (a) A polygon P with $n = 13$ vertices where $V_0(s)$ has four pockets: two pairs of dependent pockets, the sum of sizes of each pair is $\lfloor n/2 \rfloor = 6$. (b) A polygon P with $n = 13$ vertices where $V_0(s)$ has three pockets: two dependent pockets of total size $\lfloor n/2 \rfloor = 6$ and an independent pocket of size $\lfloor n/2 \rfloor = 6$.

3 Finding a Witness Point

In Section 3.1, we show that in every simple polygon P, there is a point $s \in \text{int}(P)$ that satisfies condition $\mathbf{C}_1$ In Section 3.2, we pick a point $s \in \text{int}(P)$ that satisfies condition $\mathbf{C}_1$, and move it continuously until either (i) it satisfies both conditions $\mathbf{C}_1$ and $\mathbf{C}_2$, or (ii) it becomes a double violator. In both cases, we find a witness point for Theorem 1 (by Lemmas 1, 6, and 7).

3.1 Generalized Kernel

Let P be a simple polygon with n vertices. Recall that the set of points from which the entire polygon P is visible is the *kernel*, denoted $K(P)$, which is the intersection of all halfplanes bounded by a supporting line of an edge of P and facing towards the interior of P. Lee and Preparata [16] designed an optimal $O(n)$ time algorithm for computing the kernel of a simple polygon with n vertices. We now define a generalization of the kernel. For an integer $q \in \mathbb{N}_0$, let $K_q(P)$ denote the set of points $s \in P$ such that every pocket of $V_0(s)$ has size at most q. Clearly, $K(P) = K_0(P) = K_1(P)$, and $K_q(P) \subseteq K_{q+1}(P)$ for all $q \in \mathbb{N}_0$. The set of points that satisfy condition $\mathbf{C}_1$ is $K_{\lfloor n/2 \rfloor}(P)$. For every

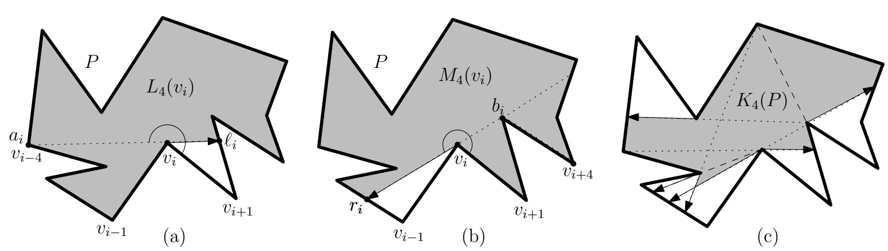

Fig. 4. (a) Polygon $L_4(v_i)$. (b) Polygon $M_4(v_i)$. (c) Polygon $K_4(P)$.

reflex vertex v, we define two polygons $L_q(v) \subseteq P$ and $M_q(v) \subseteq P$: Let $L_q(v)$ (resp. $M_q(v)$) be the set of points $s \in P$ such that v does not induce a left (resp., right) pocket of size more than q in $V_0(s)$. We have

$$K_q(P) = \bigcap_{v \text{ reflex}} (L_q(v) \cap M_q(v)) .$$

We show how to compute the polygons $L_q(v)$ and $M_q(v)$. Refer to Fig. 4. Denote the vertices of P by $(v_0, v_1, \ldots, v_{n-1})$, and use arithmetic modulo n on the indices. For a reflex vertex v_i, let $v_i a_i$ be the first edge of the shortest (geodesic) path from v_i to v_{i-q} in P. If the chord $v_i a_i$ and $v_i v_{i+1}$ meet at a reflex angle, then $v_i a_i$ is on the boundary of the *smallest* left pocket of size at least q induced by v_i (for any source $s \in P$). In this case, the ray $\overrightarrow{a_i v_i}$ enters the interior of P, and we denote by ℓ_i the first point hit on ∂P. The polygon $L_q(v_i)$ is the part of P lying on the left of the chord $\overrightarrow{v_i \ell_i}$. However, if the chord $v_i a_i$ and $v_i v_{i+1}$ meet at convex angle, then every left pocket induced by v_i has size less than q, and we have $L_q(v_i) = P$. Similarly, let $v_i b_i$ be the first edge of the shortest path from v_i to v_{i+q}. Vertex v_i can induce a right pocket of size more than q only if $b_i v_i$ and $v_i v_{i-1}$ make a reflex angle. In this case, $v_i b_i$ is the boundary of the *largest* right pocket of size at most q induced by v_i, the ray $\overrightarrow{b_i v_i}$ enters the interior of P, and hits ∂P at a point m_i, and $M_q(v_i)$ is the part of P lying on the right of the chord $\overrightarrow{v_i m_i}$. if $b_i v_i$ and $v_i v_{i-1}$ meet at a convex angle, then $M_q(v_i) = P$.

Note that every set $L_q(v_i)$ (resp., $M_q(v_i)$) is *P-convex* (a.k.a. *geodesically convex*), that is, $L_i(v_i)$ contains the shortest path between any two points in $L_q(v_i)$ with respect to P [5,11,22]. Since the intersection of P-convex polygons is P-convex, $K_q(P)$ is also P-convex for every $q \in \mathbb{N}_0$. There exists a point $s \in \text{int}(P)$ satisfying condition $\mathbf{C}_1$ iff $K_{\lfloor n/2 \rfloor}(P)$ is nonempty. We prove $K_{\lfloor n/2 \rfloor}(P) \neq \emptyset$ using a Helly-type result by Breen [8].

Theorem 2 ([8]). *Let $\mathcal{P}$ be a family of simple polygons in the plane. If every three (not necessarily distinct) members of $\mathcal{P}$ have a simply connected union and every two members of $\mathcal{P}$ have a nonempty intersection, then $\bigcap\{P : P \in \mathcal{P}\} \neq \emptyset$.*

Lemma 8. *For every simple polygon P with $n \geq 3$ vertices, $K_{\lfloor n/2 \rfloor}(P) \neq \emptyset$.*

Proof. We apply Theorem 2 for the polygons $L_{\lfloor n/2 \rfloor}(v_i)$ and $M_{\lfloor n/2 \rfloor}(v_i)$ for all reflex vertices v_i of P. By definition, $L_{\lfloor n/2 \rfloor}(v_i)$ (resp., $M_{\lfloor n/2 \rfloor}(v_i)$) is incident to at least $\lfloor n/2 \rfloor + 1$ vertices of P, namely $v_{i-\lfloor n/2 \rfloor}, \ldots, v_i$ (resp., $v_i, \ldots, v_{i+\lfloor n/2 \rfloor}$). Hence the intersection of any two sets is incident to at least at most $2(\lfloor n/2 \rfloor + 1) - n > n$ vertices of P. It remains to show that the union of any three of them is simply connected.

Suppose, to the contrary, that there are three sets whose union has a hole. Since each set is bounded by a chord of P, the hole must be a triangle bounded by the three chords on the boundary of the three polygons. Each chord is incident to a reflex vertex of P and is collinear with *another* chord of P that weakly

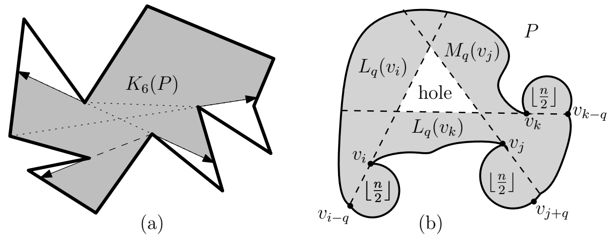

Fig. 5. (a) A simple polygon P with $n = 13$ vertices, and the generalized kernel $K_{\lfloor n/2 \rfloor}(P) = K_6(P)$. (b) A schematic picture of a triangular hole in the union of three polygons in P.

separates the vertices $\{v_i, v_{i+1}, \ldots, v_{i+\lfloor n/2 \rfloor}\}$ or $\{v_i, v_{i-1}, \ldots, v_{i-\lfloor n/2 \rfloor}\}$ from the hole. Figure 5(b) shows a schematic image. The three chords together weakly separate disjoint sets of vertices of total size at least $3\lfloor n/2 \rfloor + 3 > n$ from the hole, contradicting the fact that P has n vertices altogether. $\square$

Lemma 9. *For every $q \in \mathbb{N}_0$, $K_q(P)$ can be computed in $O(n \log n)$ time.*

3.2 Finding a Witness

In this section, we present an algorithm that, given a simple polygon P with n vertices in general position, finds a witness $s \in \text{int}(P)$ such that $\text{int}(P) \subseteq V_{\lfloor (n-2)/4 \rfloor}(s)$.

Let s_0 be an arbitrary point in $\text{int}(K_{\lfloor n/2 \rfloor}(P))$. By Lemma 8, such a point exists. We can compute the visibility polygon $V_0(s_0)$ and its pockets in $O(n)$ time [14]. The definition of $K_{\lfloor n/2 \rfloor}(P)$ ensures that s_0 satisfies condition $\mathbf{C}_1$ of Lemma 1. If it also satisfies $\mathbf{C}_2$, then $s = s_0$ is a desired witness.

Assume that s_0 does not satisfy $\mathbf{C}_2$, that is, $V_0(s_0)$ has two dependent pockets of total size at least $\lfloor n/2 \rfloor$, say a left pocket U_{ab} and (by Proposition 1) a right pocket $U_{a'b'}$. We may assume that U_{ab} is at least as large as $U_{a'b'}$, by applying a reflection if necessary, and so the size of U_{ab} is at least $\lfloor n/4 \rfloor$. Refer to Fig. 6(a). Let $c \in \partial P$ be a point sufficiently close to b such that segment bc is disjoint from all lines spanned by the vertices of P, segment s_0c is disjoint from the intersection of any two lines spanned by the vertices of P, and $s_0c \subseteq P$. In Lemma 10 (below), we find a point on segment s_0c that is a witness, or double violator, or improves a parameter (spread) that we introduce now.

For a pair of dependent pockets, a left pocket U_{ab} and (by Proposition 1) a right pocket $U_{a'b'}$, let *spread*(a, a') be the number of vertices on ∂P clockwise from a to a' (inclusive). Note that the spread is always at least the sum of the sizes of the two dependent pockets, as all vertices incident to the two pockets are counted. For a pair of pockets of total size at least $\lfloor n/2 \rfloor$, we have $\lfloor n/2 \rfloor \le \text{spread}(a, a') \le n$.

The visibility polygons of two points are combinatorially equivalent if there is a bijection between their pockets such that corresponding pockets are incident to the same sets of vertices of P. The combinatorial changes incurred by a moving point s have been thoroughly analysed in [3,4,10]. The set of points $s \in P$ that induces combinatorially equivalent visibility polygons $V_0(s)$ is a cell in the *visibility decomposition* $VD(P)$ of polygon P. It is known that each cell is convex and there are $O(n^3)$ cells, but a line segment in P intersects only $O(n)$ cells [4,9]. A combinatorial change in $V_0(s)$ occurs if s crosses a *critical line* spanned by two vertices of P, and the circular order of the rays from s to the two vertices is reversed. The possible changes are: a pocket of size 2 appears or disappears; (2) the size of a pocket increases or decreases by one; (3) two pockets merge into one pocket or a pocket splits into two pockets. Importantly, the combinatorics of $V_0(s)$ does not include the dependence between pockets: Proposition 1 will prove critical for tracking when two dependent pockets become independent.

Proposition 4. *Let $s_1 s_2$ be a line segment in* $\mathrm{int}(P)$. *Then*

(i) Every left (resp., right) pocket of $V_0(s_2)$ induced by a vertex on the left (right) of $\overrightarrow{s_1 s_2}$ is contained in a left (right) pocket of $V_0(s_1)$.
(ii) Let U_{left} and U_{right} be independent pockets of $V_0(s_1)$. Then every two pockets of $V_0(s_2)$ contained in U_{left} and U_{right}, respectively, are independent.

Lemma 10. *There is a point $s \in s_0 c$ such that one of the following holds.*

- *s satisfies both $\mathbf{C}_1$ and $\mathbf{C}_2$;*
- *s is a double violator;*
- *s satisfies $\mathbf{C}_1$ but violates $\mathbf{C}_2$ due to two pockets of spread* $\leq \mathrm{spread}(a, a') - \lfloor n/4 \rfloor$.

Proof. We move a point $s \in s_0 c$ from s_0 to c and trace the combinatorial changes of the pockets of $V_0(s)$, and their dependencies. Initially, when $s = s_0$, all pockets have size at most $\lfloor n/2 \rfloor - 1$; and there are two dependent pockets, a left pocket U_{ab} on the left of $\overrightarrow{s_0 c}$ and, by Proposition 1, a right pocket $U_{a'b'}$ on the right of $\overrightarrow{s_0 c}$, of total size at least $\lfloor n/2 \rfloor$. When $s = c$, every left pocket of $V_0(s)$ on the left of $\overrightarrow{s_0 c}$ is independent of any right pocket on the right of $\overrightarrow{s_0 c}$.

Consequently, when s moves from s_0 to c, there is a critical change from $s = s_1$ to $s = s_2$ such that $V_0(s_1)$ still has two dependent pockets of size at least $\lfloor n/2 \rfloor$ where the left (resp., right) pocket is on the left (right) of $\overrightarrow{s_0 c}$; but $V_0(s_2)$ has no two such pockets. (See Fig. 6 for examples.) Let U_{left} and U_{right} denote the two violator pockets of $V_0(s_1)$. The critical point is either a combinatorial change (i.e., the size of one of these pockets drops), or the two pockets become independent. By Proposition 4, we have $U_{\mathrm{left}} \subseteq U_{ab}$ and $U_{\mathrm{right}} \subseteq P \setminus U_{ab}$, and the spread of U_{left} and U_{right} is at most $\mathrm{spread}(a, a')$. We show that one of the statements in Lemma 10 holds for s_1 or s_2.

If s_2 satisfies both $\mathbf{C}_1$ and $\mathbf{C}_2$, the our proof is complete (Fig. 6(a-b)). If s_2 violates $\mathbf{C}_1$, i.e., $V_0(s_2)$ has a pocket of size $\geq \lfloor n/2 \rfloor$, then $V_0(s_1)$ also has a combinatorially equivalent pocket independent of U_{left} and U_{right}, and so s_1 is a double violator. Finally, if s_2 violates $\mathbf{C}_2$, i.e., $V_0(s_2)$ has two dependent pockets of total size $\lfloor n/2 \rfloor$, then the left pocket of this pair is not contained in U_{ab}. We

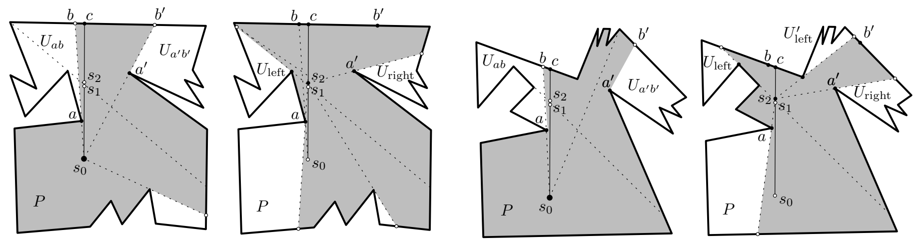

Fig. 6. (a) A polygon with $n = 21$ vertices where s_0 violates $\mathbf{C}_2$ a pair of dependent pockets U_{ab} and $U_{a'b'}$. (b) Point $s_2 \in s_0c$ satisfies both $\mathbf{C}_1$ and $\mathbf{C}_2$. (c) A polygon with $n = 21$ vertices where s_0 violates $\mathbf{C}_2$ with a pair of pockets U_{ab} and $U_{a'b'}$ of spread 19. (d) Point s_2 also violates $\mathbf{C}_2$ with a pair of pockets of spread 13.

have two subcases to consider: (i) If the right pocket of this new pair is contained in U_{right}, then their spread is at most spread$(a, a') - \lfloor n/4 \rfloor$ (Fig. 6(c-d)). (ii) If the right pocket of the new pair is disjoint from U_{right}, then $V_0(s_1)$ also has a combinatorially equivalent pair of pockets, which is different from U_{left} and U_{right}, and so s_1 is a double violator. □

Lemma 11. *A point $s \in s_0c$ described in Lemma 10 can be found in $O(n \log n)$ time.*

Proof. It is enough to show that the critical positions, s_1 and s_2, in the proof of Lemma 10 can be computed in $O(n \log n)$ time. We use the data structure of Chen and Daescu [9], which is constructed by decomposing s_0c into a set of $O(n)$ intervals with combinatorially distinct region and recording the changing region in a persistent search tree.

However, the data structure of [9] only stores the visible region, not whether the region induces dependent pockets. The main technical difficulty is that $\Omega(n^2)$ dependent pairs might become independent as s moves along s_0s (even if we consider only pairs of total size at least $\lfloor n/2 \rfloor$), in contrast to only $O(n)$ combinatorial changes of the visibility region. We reduce the number of relevant events by focusing on only the "large" pockets (pockets of size at least $\lfloor n/4 \rfloor$), and maintaining at most one pair that violates $\mathbf{C}_2$ for each large pocket. (In a dependent pair of size $\geq \lfloor n/2 \rfloor$, one of the pockets has size $\geq \lfloor n/4 \rfloor$.)

We augment the data structure of [9] as follows. We maintain the list of all left (resp., right) pockets of $V_0(s)$ lying on the left (right) of $\overrightarrow{s_0c}$, sorted in counterclockwise order along ∂P. We also maintain the set of *large* pockets of size at least $\lfloor n/4 \rfloor$ from these two lists. There are at most 4 large pockets for any $s \in s_0c$. For a large pocket $U_{\alpha\beta}$ of $s \in s_0c$, we maintain one possible other pocket $U_{\alpha'\beta'}$ of $V_0(s)$ such that they together violate $\mathbf{C}_2$. If there are several such pockets $U_{\alpha'\beta'}$, we maintain only the one where α' (the reflex vertex that induces $U_{\alpha'\beta'}$) is farthest from c along ∂P. Thus, we maintain a set $\mathcal{U}(s)$ of at most 4 pairs $(U_{\alpha\beta}, U_{\alpha'\beta'})$. Finally, for each of pair $(U_{\alpha\beta}, U_{\alpha'\beta'}) \in \mathcal{U}$, we maintain the positions $s' = sc \cap \alpha\alpha'$ where the pair $(U_{\alpha\beta}, U_{\alpha'\beta'})$ becomes independent assuming that neither $U_{\alpha\beta}$ nor $U_{\alpha'\beta'}$ goes through combinatorially before s reaches s'. We use [9], combined with these supplemental structures, to find critical points $s_1, s_2 \in s_0c$ such that $\mathcal{U}(s_1) \neq \emptyset$ but $\mathcal{U}(s_2) = \emptyset$.

We still need to show that $\mathcal{U}(s)$ can be maintained in $O(n \log n)$ time as s moves from s_0 to c. A pair $(U_{\alpha\beta}, U_{\alpha'\beta'})$ has to be updated if $U_{\alpha\beta}$ or $U_{\alpha'\beta'}$ undergoes a combinatorial change, or if they become independent (i.e., $s \in \alpha\alpha'$). Each large pocket undergoes $O(n)$ combinatorial changes affect them by Proposition 4, and there are $O(n)$ reflex vertices along ∂P between a and a' (these are the candidates for α'). No update is necessary when β or β' changes but $U_{\alpha\beta}$ remains large and the total size of the pair is at least $\lfloor n/2 \rfloor$. If the size of $U_{\alpha\beta}$ drops below $\lfloor n/4 \rfloor$, we can permanently eliminate the pair from $\mathcal{U}$. In all other cases, we search for a new vertex α', by testing the reflex vertices that induce pockets from the current α' towards c along ∂P until we either find a new pocket $U_{\alpha'\beta'}$ or determine that $U_{\alpha\beta}$ is not dependent of any other pocket with joint size $\geq \lfloor n/2 \rfloor$. We can test dependence between $U_{\alpha\beta}$ and a candidate for $U_{\alpha'\beta'}$ in $O(\log n)$ time (test $\alpha\alpha' \subseteq P$ by a ray shooting query). Each update of $(U_{\alpha\beta}, U_{\alpha'\beta'})$ decreases the size of the large pocket $U_{\alpha\beta}$ or moves the vertex α' closer to c. Therefore, we need to test dependence between only $O(n)$ candidate pairs of pockets. Overall, the updates to $\mathcal{U}(s)$ take $O(n \log n)$ time. □

We are now ready to prove Theorem 1.

Proof (of Theorem 1). Let P be a simple polygon with $n \geq 3$ vertices. Compute the generalized kernel $K_{\lfloor n/2 \rfloor}(P)$, and pick an arbitrary point $s_0 \in \text{int}(K_{\lfloor n/2 \rfloor}(P))$, which satisfies $\mathbf{C}_1$. If s_0 also satisfies $\mathbf{C}_2$, then $\text{int}(P) \subseteq V_{\lfloor (n-2)/4 \rfloor}(s_0)$ by Lemma 1. Otherwise, there is a pair of dependent pockets, U_{ab} and $U_{a'b'}$, of total size at least $\lfloor n/2 \rfloor$ and $\lfloor n/2 \rfloor \leq \text{spread}(a, a') \leq n$. Invoke Lemma 10 up to four times to find a point $s \in \text{int}(P)$ that either satisfies both $\mathbf{C}_1$ and $\mathbf{C}_2$, or is a double violator. If s satisfies $\mathbf{C}_1$ and $\mathbf{C}_2$ then Lemma 1 completes the proof. If s is a double violator, apply Lemma 6 or Lemma 7 as appropriate to complete the proof. The overall running time of the algorithm is $O(n \log n)$ from the combination of Lemmas 6, 7, 9, and 11.

For every $k \geq 1$, the diffuse reflection diameter of the zig-zag polygon (cf. Fig. 1) with $n = 4k + 2$ vertices is $k = \lfloor (n-2)/4 \rfloor$. Adding up to 3 dummy vertices on the boundary of a zig-zag polygon gives n-vertex polygons P_n with $R(P_n) = \lfloor (n-2)/4 \rfloor$ for all $n \geq 6$. Finally, every simple polygon with $3 \leq n \leq 5$ vertices is star-shaped, and so its diffuse reflection radius is $0 = \lfloor (n-2)/4 \rfloor$. □

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