Face guards for art galleries

Diane L. Souvaine[1], **Raoul Veroy, and Andrew Winslow**[1]

Tufts University, Medford, MA, USA
dls@cs.tufts.edu, rveroy@cs.tufts.edu, awinslow@cs.tufts.edu

Abstract. A classic problem in computational geometry is the art gallery problem: given an enclosure, how should guards be placed to ensure every location in the enclosure is seen by some guard. In this paper we consider guarding the interior of a simple polyhedron using *face guards*: guards that roam over an entire interior face of the polyhedron. Bounds for the number of face guards g that are necessary and sufficient to guard any polyhedron with f faces are given. We show that for orthogonal polyhedra, $\lfloor f/7 \rfloor \leq g \leq \lfloor f/6 \rfloor$, while for general polyhedra $\lfloor f/5 - 2/5 \rfloor \leq g \leq \lfloor f/2 \rfloor$.

Introduction

In computational geometry, few problems are as recognizable as art gallery problems: given a region and a choice of how to place guards in the region, determine the locations of guards in order to see all locations in the region. In 2D, such problems have been considered for many years (see the surveys in [**2**], [**3**] and [**5**]), however work in 3D is less extensive. Grünbaum and O'Rourke [**2**] and Szabó and Talata [**4**] consider guarding the exterior of a polyhedron with *vertex guards* stationed at vertices of the polyhedron, while Bose *et al.* [**1**] and Urrutia [**5**] consider guarding the interior of a polyhedron with *edge guards* free to walk along an entire edge.

In this work we consider an entirely new type of guard called a *face guard* that is free to move about a face (including its edges), and guards any location visible from *some* point on the face. We bound the number of face guards g sufficient to guard orthogonal and general simple polyhedra with f faces. In the orthogonal case we show that $\lfloor f/7 \rfloor \leq g \leq \lfloor f/6 \rfloor$, while in the general case $\lfloor f/5 - 2/5 \rfloor \leq g \leq \lfloor f/2 \rfloor$.

1 Definitions

In this paper we consider polyhedra with genus 0 (i.e. homeomorphic to a sphere) with faces that are *not* necessarily simply-connected (homeomorphic to discs). In contrast to some work on 3D art gallery problems, we consider guarding the interior of a polyhedron, not its exterior. A polyhedron is guarded by selecting interior faces to be face guards.

We say a point p in the interior of the polyhedron is *seen* by a face guard if the open line segment connecting p to some point on the closed face does not intersect the boundary of the polyhedron. Thus the region *guarded* by a face guard is the set of all points that are seen by a point on the face.

2 Orthogonal polyhedra

We start by considering the class of *orthogonal polyhedra*: polyhedra with every edge parallel to one of the three axes. By definition, each face of an orthogonal polyhedron has

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a normal vector parallel to one of the three axes. Thus, the interior faces of orthogonal polyhedra can be partitioned into six sets according to the directions of their normal vectors.

Lemma 2.1 *Let F be the set of all interior faces of an orthgonal polyhedron with normal vectors in the same direction. Then F is sufficient to guard the polyhedron.*

Proof. Without loss of generality, let F be the set of faces with normal vectors pointing in the positive x-direction. Let p be a point in the interior of the polyhedron. Consider extending a ray from p in the negative x-direction until it intersects a face f of the polyhedron. This face f must have a normal vector in the positive x-direction and sees p. So the set F guards the entire polyhedron. □

Lemma 2.2 *Let P be a polyhedron with f faces. Then $\lfloor f/6 \rfloor$ face guards are sufficient to guard any orthogonal polyhedron.*

Proof. The normal vectors of the interior faces of P partition these faces into six sets. By Lemma 2.1, each of these six sets are sufficient to guard P. By the pigeonhole principle, at least one of these sets has size at most $\lfloor f/6 \rfloor$. □

Lemma 2.3 *For all $f = 21k$ where k is a positive integer, there exist orthogonal polyhedra with f faces that require $3k$ face guards.*

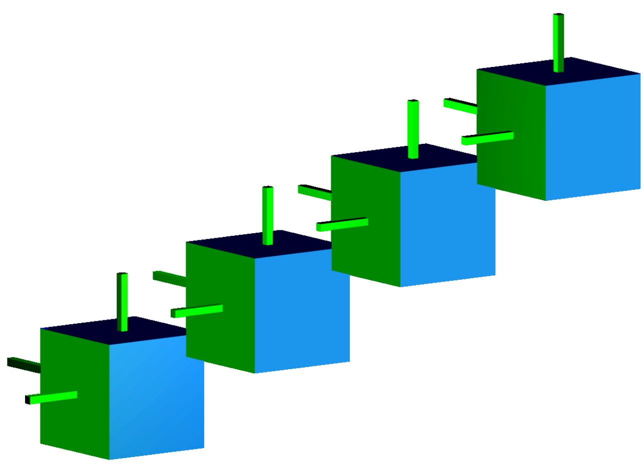

FIGURE 1. An orthogonal polyhedron with $21k$ faces requiring $3k$ guards for $k = 4$.

Proof. The proof is by explicit construction as seen in Figure 1. We use k large cubes, each with 3 narrow 'chimneys' attached to the cube's front, left, and top faces. The large cubes are attached to each other at their corners to form a long chain. Each cube and its 3 chimneys have a total of 21 faces, so the entire construction has $21k$ faces.

Consider using face guards to guard the interior of the polyhedron. We claim that a distinct face guard is needed for each chimney. This can be seen by considering the set of faces that see a point p deep in a chimney. Certainly each face of the chimney and the face of the cube containing can see p. Additionally, the face of the cube opposite of the face containing the chimney can also see p. Because the narrowness and length of the chimney, no other faces can see p, including those from adjacent cubes. Intuitively, one can think of placing a light at p, and the light leaving the chimney as a focused beam which strikes only the center of the opposite face of the cube. Applying this analysis to all three chimneys in a cube, we see that a distinct face guard is needed for each chimney, thus guarding the entire construction requires $3k$ face guards. □

Theorem 2.4 *Let g be the minimum number of face guards sufficient to guard any orthogonal polyhedron with f faces. Then $\lfloor f/7 \rfloor \le g \le \lfloor f/6 \rfloor$.*

Proof. Combining the results from Lemmas 3.1 and 3.2 gives the inequalities. □

3 General polyhedra

In this section we consider guarding general simple polyhedra. In comparison to orthogonal polyhedra, we find that the necessary and sufficient numbers of guards are both increased.

Lemma 3.1 *Let P be a polyhedron with f faces. Then $\lfloor f/2 \rfloor$ face guards are sufficient to guard P.*

Proof. This proof is similar to the proof of Lemma 2.2. Consider the dot product of the normal vectors of the interior faces of P with a vector in the positive x-direction. Each dot product is either negative or non-negative. Partition the faces into two sets according to the values of their dot products. One of these sets must have size at most $\lfloor f/2 \rfloor$. Without loss of generality, suppose the set of faces with non-negative dot products has size at most $\lfloor f/2 \rfloor$, and call this set F.

Let p be a point in the interior of P. Consider extending a ray from p in negative x-direction. The first face intersected must be in F, and so F guards p. So F is sufficient to guard P. □

Lemma 3.2 *For all $f = 5k + 2$ where k is a positive integer, there exist polyhedra with f faces that require $\frac{1}{5}f - \frac{2}{5}$ face guards.*

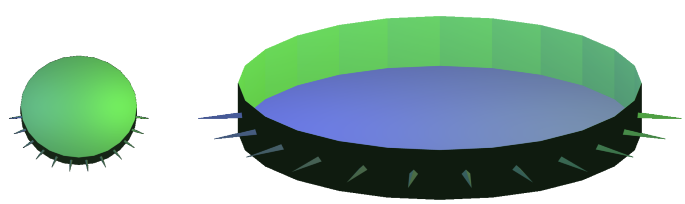

FIGURE 2. A polyhedron with $5k + 2$ faces requiring k face guards for $k = 12$. At left is the complete polyhedron, while at right is a larger version of the polyhedron with the top face removed.

Proof. The proof is by explicit construction (see Figure 2). The construction is a large disc-shaped polyhedron consisting of two large regular faces with $2k$ edges each, and $2k$ small faces connecting them. Half (k) of the small faces have a narrow *spike* consisting of three faces extending out of the polyhedron meeting at a single point. The tip of the spike can only see a single small face on the opposite side of the polyhedron. The face it sees does not have a spike leaving it. The construction has a total of $2k+2+3k = 5k+2$ faces.

We claim that the vertex at the pointy end of each spike can only been seen by one of 5 faces: the three faces creating the spike, the face the spike leaves, and the face on the far side of the polyhedron that the pointy end sees. Looking at the sets of faces that see the pointy end of each spike, we notice they are disjoint. So a distinct face guard is needed for the pointy end of each spike. There are k such spikes, so k face guards are needed. □

Theorem 3.3 *Let g be the number of face guards sufficient to guard any polyhedron with f faces. Then $\left\lfloor \frac{1}{5}f - \frac{2}{5} \right\rfloor \leq g \leq \lfloor f/2 \rfloor$.*

Proof. Combining Lemmas 3.1 and 3.2 gives the inequalities. □

4 Conclusion and open problems

In this work we have introduced the notion of face guards for 3D art gallery problems. This new type of guard is permitted to walk about freely on a closed interior face, and guards any point in the polyhedron seen from some location on this face. We give bounds on the number of face guards needed to guard the interior of both orthogonal and general simple polyhedra.

This new type of guard for polyhedra suggests an interesting set of open problems. Such problems include the complexity of minimizing the number of face guards, necessary and sufficient numbers of guards for tetrahedralizable, surface triangulated, and non-zero genus polyhedra. We also support the investigation of *open face guards*, in which the boundary of the face is omitted.

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