CSCI 3310 Solutions: Proofs

1 Direct Proof

Prove each of the following theorems using direct proof.

**Theorem 1.** Let $n \in \mathbb{Z}$. If $n$ is odd, then $n^3 + 3$ is even.

**Proof.** Let $n$ be an odd integer. So there exists $i \in \mathbb{Z}$ such that $n = 2i + 1$. So

\[
\begin{align*}
 n^3 + 3 &= (2i + 1)^3 + 3 \\
 &= (2i + 1)(4i^2 + 4i + 1) + 3 \\
 &= 8i^3 + 8i^2 + 2i + 4i^2 + 4i + 1 + 3 \\
 &= 8i^3 + 12i^2 + 6i + 4 \\
 &= 2(4i^3 + 6i^2 + 3i + 2)
\end{align*}
\]

Since $i \in \mathbb{Z}$, $4i^3 + 6i^2 + 3i + 2 = j \in \mathbb{Z}$. So $n^3 + 3 = 2j$ with $j \in \mathbb{Z}$.

**Theorem 2.** Let $a, b \in \mathbb{Z}$. If $a, b$ are odd, then $ab$ is odd.

**Proof.** Let $a$ and $b$ be odd integers. So there exist $i, j \in \mathbb{Z}$ such that $a = 2i + 1$ and $b = 2j + 1$. So $ab = (2i + 1)(2j + 1) = 4ij + 2i + 2j + 1 = 2(2ij + i + j) + 1$. Since $i, j \in \mathbb{Z}$, $2ij + i + j = k \in \mathbb{Z}$. So $ab = 2k + 1$ with $k \in \mathbb{Z}$. So $ab$ is odd.

**Theorem 3.** For all $n \in \mathbb{Z}$, $(289 \mid n) \Rightarrow (17 \mid n)$.

**Proof.** Let $n$ be an integer divisible by 289. Then there exists $i \in \mathbb{Z}$ such that $n = 289i$. So $n = 17 \cdot 17 \cdot i$. Since $i \in \mathbb{Z}$, $17i = j \in \mathbb{Z}$. So $n = 17j$ with $j \in \mathbb{Z}$. So $17 \mid n$.

**Theorem 4.** Let $A, B, C$ be sets. If $A \subseteq C$ and $B \subseteq C$, then $(A \cap B) \subseteq (A \cap C)$.

**Proof.** Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. Since $x \in B$ and $B \subseteq C$, $x \in C$. So $x \in A$ and $x \in C$. Thus $x \in A \cap C$. So if $x \in A \cap B$, then $x \in A \cap C$. So $(A \cap B) \subseteq (A \cap C)$.
Theorem 5. If $a, b \in \mathbb{R}$, then $a^2 + b^2 \geq 2ab$.

Proof. Let $a$ and $b$ be real numbers.
Then $(a - b)^2 \geq 0$.
So $a^2 - 2ab + (-b)^2 \geq 0$.
So $a^2 + b^2 \geq 2ab$. \hfill \qed

2 Proof by Cases

Prove each of the following theorems using proof by cases.

Theorem 6. For any integer $n$, $n^2 + 3n + 3$ is odd.

Proof. By cases.
An integer $n$ is even or odd.

If $n$ is even, then there exists $i \in \mathbb{Z}$ such that $n = 2i$.
So

$$n^2 + 3n + 3 = (2i)^2 + 3(2i) + 3$$
$$= 4i^2 + 6i + 3$$
$$= 4i^2 + 6i + 2 + 1$$
$$= 2(2i^2 + 3i + 1) + 1$$

Since $i \in \mathbb{Z}$, $2i^2 + 3i + 1 = j \in \mathbb{Z}$.
So $n^2 + 3n + 3 = 2j + 1$ with $j \in \mathbb{Z}$.
So $n^2 + 3n + 3$ is odd.

If $n$ is odd, then there exists $i \in \mathbb{Z}$ such that $n = 2i + 1$.
So

$$n^2 + 3n + 3 = (2i + 1)^2 + 3(2i + 1) + 3$$
$$= (4i^2 + 4i + 1) + (6i + 3) + 3$$
$$= 4i^2 + 10i + 4 + 3$$
$$= 4i^2 + 10i + 6 + 1$$
$$= 2(2i^2 + 5i + 3) + 1$$

Since $i \in \mathbb{Z}$, $2i^2 + 5i + 3 = j \in \mathbb{Z}$.
So $n^2 + 3n + 3 = 2j + 1$ with $j \in \mathbb{Z}$.
So $n^2 + 3n + 3$ is odd. \hfill \qed

Theorem 7. For any integer $n$, $n^4 - n^3 \geq 0$. 

Proof. By cases.
Either $n \geq 1$, $n = 0$, or $n \leq -1$.

Suppose $n \geq 1$. Then

\[
\begin{align*}
   n &\geq 1 \\
   n \cdot n &\geq n \cdot 1 \\
   n^2 &\geq n \\
   n^2 \cdot n^2 &\geq n^2 \cdot n \\
   n^4 &\geq n^3 \\
   n^4 - n^3 &\geq 0
\end{align*}
\]

Suppose $n = 0$. Then $n^4 - n^3 = 0^4 - 0^3 = 0 - 0 = 0 \geq 0$.

Suppose $n \leq -1$. Then $n = -1 \cdot i$ for some $i \in \mathbb{N}$.
So $n^4 = (-1 \cdot i)^4 = (-1)^4 \cdot i^4 = i^4$.
Also, $n^3 = (-1 \cdot i)^3 = (-1)^3 \cdot i^3 = -i^3$.
Since $i \in \mathbb{N}$, $n^4 = i^4 \geq 1$ and $n^3 = -i^3 < 0$.
So $n^4 \geq 1 > 0 > -i^3 = n^3$ and thus $n^4 \geq n^3$.
So $n^4 - n^3 \geq 0$.

\[\square\]

**Theorem 8.** Let $A$, $B$, and $C$ be sets. If $A \subseteq C \land B \subseteq C$, then $A \cup B \subseteq C$.

**Proof.** By cases.
An element $x \in A \cup B$ is either in $A$ or $B$.

Suppose $x \in A$.
If $A \subseteq C \land B \subseteq C$, then $A \subseteq C$.
If $x \in A$ and $A \subseteq C$, then $x \in C$.

Suppose $x \in B$.
If $A \subseteq C \land B \subseteq C$, then $B \subseteq C$.
If $B \subseteq C$, then $x \in C$.

So if $A \subseteq C \land B \subseteq C$ and $x \in A \cup B$, then $x \in C$.

\[\square\]

3 Proof by Contrapositive

Prove each of the following theorems using proof by contrapositive.

**Theorem 9.** Let $n$ be an integer. If $n^3$ is odd, then $n$ is odd.
Proof. By contrapositive.
Assume $n$ is not odd.
So $n$ is even.
Then $\exists a \in \mathbb{Z}, n = 2a$. So $n^3 = (2a)^3 = 8a^3 = 2(4a^3)$.
Let $b = 4a^3 \in \mathbb{Z}$.
Then $\exists b \in \mathbb{Z}, n^3 = 2b$.
So $n^3$ is even.
So $n^3$ is not odd.

Theorem 10. Let $a, b, c$ be real numbers. If $a + b + c$ is negative, then $a, b, c$ is negative.

Proof. By contrapositive.
Let $a, b, c \in \mathbb{R}$ with $a, b, c \geq 0$.
Then $a + b + c \geq 0 + 0 + 0 = 0$.
That is, $a + b + c$ is not negative.

Theorem 11. Let $x, y \in \mathbb{Z}$. If $x - y$ is odd, then $x$ is odd or $y$ is odd.

Proof. By contraposition.
Let $x, y \in \mathbb{Z}$ with $x$ and $y$ even.
Then there exist $i, j \in \mathbb{Z}$ such that $x = 2i$ and $y = 2j$.
So $x - y = 2i - 2j = 2(i - j)$.
Since $i, j \in \mathbb{Z}, i - j = k \in \mathbb{Z}$.
So $x - y = 2k$ with $k \in \mathbb{Z}$.
So $x - y$ is even.

4 Proof by Contradiction

Prove each of the following theorems using proof by contradiction.

Theorem 12. For every $x \in \mathbb{Q}^+$, there exists $y \in \mathbb{Q}^+$ with $y < x$.

Proof. By contradiction.
Suppose there exists $x \in \mathbb{Q}^+$ such that $\forall y \in \mathbb{Q}^+, y \geq x$.
Since $x \in \mathbb{Q}^+, x = p/q$ with $p, q \in \mathbb{N}$.
Then $r = 2q = \mathbb{N}$.
Consider $y = p/r = p/(2q) = x/2$.
Since $p, r \in \mathbb{N}, y \in \mathbb{Q}^+$.
Since $1 > 1/2$ and $x > 0, x > x/2 = y$.
So $\exists y \in \mathbb{Q}^+$ with $y < x$.

Theorem 13. $\sqrt{2}$ is not rational. (Hint: use Theorem 9.)

Proof. By contradiction.
Suppose $\sqrt{2}$ is rational.
Then $\exists p \in \mathbb{Z}, q \in \mathbb{N}$ such that $\sqrt{2} = p/q$ and $\gcd(p, q) = 1$.  


So \( 2 = p^3/q^3 \).
So \( 2q^3 = p^3 \).
So \( p^3 \) is even.
Then by Theorem 9, \( p \) is even.
So there exists \( i \in \mathbb{Z} \) such that \( p = 2i \).
So \( 2q^3 = (2i)^3 = 8i^3 \).
So \( q^3 = 4i^3 = 2(2i^3) \) with \( i \in \mathbb{Z} \).
So \( q^3 \) is even.
Then by Theorem 9, \( q \) is even.
So \( p \) and \( q \) are even.
So \( \gcd(p, q) \geq 2 \).

**Theorem 14.** Let \( n \) and \( m \) be even integers. Then \( n^3 - 2m - 2 \neq 0 \). (Hint: use Theorem 9.)

**Proof.** By contradiction.
Suppose \( n^3 - 2m - 2 = 0 \).
Then \( n^3 = 2m + 2 = 2(m + 1) \).
Let \( a = m + 1 \in \mathbb{Z} \).
Then \( \exists a \in \mathbb{Z} \) such that \( n^3 = 2a \).
So \( n^3 \) is even.
Then by Theorem 9, \( n \) is even.
So \( \exists b \in \mathbb{Z} \) such that \( n = 2b \).
So \( 2(4b^3) = 8b^3 = (2b)^3 = n^3 = 2m + 2 = 2(m + 1) \).
So \( 4b^3 = m + 1 \) and \( 4b^3 - 1 = m \).
Then \( 2(2b^3 - 2) + 1 = m \).
Let \( c = 2b^3 - 2 \in \mathbb{Z} \).
Then \( 2c + 1 = m \) with \( c \in \mathbb{Z} \), so \( m \) is odd.

\[ \square \]