CSCI 3310 Solutions: Functions

1 Domain, Codomain, Image

Problem 1. Let \( f : \mathbb{Z} \to \mathbb{Z}^+ \) with \( f(n) = |n| + 5 \). What are the domain, codomain, and image of \( f \)?

Solution 1.
- **Domain:** \( \mathbb{Z} \).
- **Codomain:** \( \mathbb{Z}^+ \).
- **Image:** \( \{x \in \mathbb{Z} : x \geq 5\} \).

Problem 2. Let \( f : \mathbb{R} \to \mathbb{Z} \) with \( f(x) = \lfloor x/10 \rfloor \). What are the domain, codomain, and image of \( f \)?

Solution 2.
- **Domain:** \( \mathbb{R} \).
- **Codomain:** \( \mathbb{Z} \).
- **Image:** \( \mathbb{Z} \).

Problem 3. Let \( f : \mathbb{Q} \to \mathbb{R} \) with \( f(x) = x^2 \). What are the domain, codomain, and image of \( f \)?

Solution 3.
- **Domain:** \( \mathbb{Q} \).
- **Codomain:** \( \mathbb{R} \).
- **Image:** \( \{x \in \mathbb{Q} : x \geq 0\} \).

2 Function Properties and Common Functions

Problem 4. Let \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) with \( f(x) = x/3 \).

- Is \( f \) injective?
- Is \( f \) surjective?
- Is \( f \) bijective?
- Is \( f^{-1} \) a function?
- Is \( f \) increasing?
• Is $f$ strictly increasing?

Solution 4.

• Yes, injective.
• Yes, surjective.
• Yes, bijective.
• Yes, $f^{-1}$ is a function ($f^{-1}(x) = 3x$).
• Yes, $f$ is increasing.
• Yes, $f$ is strictly increasing.

Problem 5. Let $f : \mathbb{R} \to \mathbb{Z}$ with $f(x) = 3\lfloor x \rfloor$.

• Is $f$ injective?
• Is $f$ surjective?
• Is $f$ bijective?
• Is $f^{-1}$ a function?
• Is $f$ increasing?
• Is $f$ strictly increasing?

Solution 5.

• No, not injective ($f(0) = 0 = f(0.5)$).
• No, not surjective (no $x \in \mathbb{R}$ such that $f(x) = 1$).
• No, not bijective (not injective, not surjective).
• No, $f^{-1}$ is not a function (since $f$ is not bijective).
• Yes, $f$ is increasing.
• No, $f$ is not strictly increasing.

Problem 6. Let $f : \mathbb{N} \to \mathbb{R}$ with $f(n) = \log_2(n)$.

• Is $f$ injective?
• Is $f$ surjective?
• Is $f$ bijective?
• Is $f^{-1}$ a function?
• Is \( f \) increasing?
• Is \( f \) strictly increasing?

Solution 6.
• Yes, injective.
• No, not surjective (no \( n \in \mathbb{N} \) such that \( f(n) = 0 \)).
• No, not bijective (not surjective).
• No, \( f^{-1} \) is not a function (since \( f \) is not bijective).
• Yes, \( f \) is increasing.
• Yes, \( f \) is strictly increasing.

Problem 7. Let \( f : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) with \( f(x) = x^{\lfloor x \rfloor} \).
• Is \( f \) injective?
• Is \( f \) surjective?
• Is \( f \) bijective?
• Is \( f^{-1} \) a function?
• Is \( f \) increasing?
• Is \( f \) strictly increasing?

Solution 7.
• No, not injective (\( f(0.5) = 1 = f(0.6) \)).
• No, not surjective (no \( x \in \mathbb{N} \) such that \( f(x) = 0.5 \)).
• No, not bijective (not injective, not surjective).
• No, \( f^{-1} \) is not a function (since \( f \) is not bijective).
• Yes, \( f \) is increasing.
• No, \( f \) is not strictly increasing (\( f(0.5) = 1 = f(0.6) \)).
3 Function Composition

Problem 8. Let $f : \mathbb{R}^+ \to \mathbb{N}$ with $f(x) = [(x + 1)^2]$ and $g : \mathbb{N} \to \mathbb{R}^+$ with $g(n) = \frac{4n}{5}$.

- Is $f \circ g$ a function? If so, write the function.
- Is $g \circ f$ a function? If so, write the function.

Solution 8.

$f \circ g : \mathbb{N} \to \mathbb{N}$ is the function $(f \circ g)(n) = [(4n/5 + 1)^2]$.

$g \circ f : \mathbb{R}^+ \to \mathbb{R}^+$ is the function $(g \circ f)(x) = 4[(x + 1)^2]/5$.

Problem 9. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ with $f(x) = x/2$ and $g : \mathbb{R}^+ \to \mathbb{N}$ with $g(x) = [x + 1]$.

- Is $f \circ g$ a function? If so, write the function.
- Is $g \circ f$ a function? If so, write the function.

Solution 9.

$f \circ g$ is not a function (codomain of $g$ is $\mathbb{N}$, domain of $f$ is $\mathbb{R}^+$).

$g \circ f : \mathbb{R}^+ \to \mathbb{N}$ is the function $(g \circ f)(x) = [x/2 + 1]$.

4 Asymptotic Notation

Problem 10. Replace $\bigcirc$ with $O$, $\Omega$, or $\Theta$ in the following expressions (use $\Theta$ if possible):

$4n^4 = O(2n^2)$, $1 = O(100)$, $3n = O(n^2)$, $32 = O(2^n)$, $31n^3 = O(n^{30})$.

Solution 10. $4n^4 = \Omega(2n^2)$, $1 = \Theta(100)$, $3n = O(n^2)$, $32 = O(2^n)$, $31n^3 = O(n^{30})$.

Problem 11. Replace $\bigcirc$ with $O$, $\Omega$, or $\Theta$ in the following expressions (use $\Theta$ if possible):

$2n^2 = \Theta(2^n)$, $\log(n) = \Theta(50)$, $n \log(n) = \Theta(n^{1.1})$, $\sqrt{n} = \Theta(2^n)$, $16 \cdot 4^n = \Theta(3^n)$.

Solution 11. $2n^2 = \Omega(2^n)$, $\log(n) = \Omega(50)$, $n \log(n) = O(n^{1.1})$, $\sqrt{n} = O(2^n)$, $16 \cdot 4^n = \Omega(3^n)$

Problem 12. Replace $\bigcirc$ with $O$, $\Omega$, or $\Theta$ in the following expressions (use $\Theta$ if possible):

$\log(n) = O([\log(n)])$, $n^{\log(n)} = \Theta(2^n)$, $(\log(n))^n = \Theta(2^n)$, $2^n = O(n^n)$.

Solution 12. $\log(n) = \Theta([\log(n)])$, $n^{\log(n)} = \Theta(2^n)$, $(\log(n))^n = \Omega(2^n)$, $2^n = O(n^n)$

Problem 13. Prove that $100n = O(n^2)$.

Solution 13. Let $n_0 = 1$ and $c = 100$. Then for all $n \geq n_0 = 1$:

\[
100n \leq 100n \cdot 1 \\
\leq 100n \cdot n \text{ (since } n \geq 1) \\
\leq 100n^2 \\
\leq cn^2
\]

So there exists $c \in \mathbb{R}$, $n \in \mathbb{N}$ such that for all $n \geq n_0$, $100n \leq cn^2$.  

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Problem 14. Prove that $100n = O(n \log_2(n))$.

Solution 14. Let $n_0 = 2^{100}$ and $c = 1$. Then for all $n \geq n_0 = 2^{100}$:

\[
2^{100} \leq n \\
100 \leq \log_2(n) \text{ (since both sides are positive)} \\
100n \leq \log_2(n) \cdot n \text{ (since } n \geq 2^{100} > 0) \\
\leq n \log_2(n) \\
\leq cn \log_2(n)
\]

So there exists $c \in \mathbb{R}$, $n \in \mathbb{N}$ such that for all $n \geq n_0$, $100n \leq cn \log_2 n$.

Problem 15. Prove that $50n \log_{10}(n) = \Theta(n \log_2(n))$.

Solution 15. Recall that there’s an identity that says $\log_{10}(n) = \log_2(n) / \log_2 10$ for all $n \in \mathbb{R}$. Let $n_0 = 1$ and $c = 50 / \log_2 10$. Then:

\[
\log_{10}(n) = \frac{\log_2(n)}{\log_2 10} \\
= \log_2(n) \cdot \frac{c}{50} \\
50 \log_{10}(n) = \log_2(n) \cdot c \\
50 \log_{10}(n) = c \log_2(n) \\
50n \log_{10}(n) = cn \log_2(n)
\]

So there exists $c_1, c_2 \in \mathbb{R}^+$ (both equal to $50 / \log_2 10$) and $n_0 \in \mathbb{N}$ such that $c_1 n \log_2(n) \leq 50n \log_{10}(n) \leq c_2 n \log_2(n)$ for all $n \geq n_0$. 