CSCI 3310 Solutions: Counting

1 Product and Sum Rules

Problem 1. How many length-5 strings over the alphabet of lowercase letters, uppercase letters, and numbers exist?

Solution 1. There are $26 + 26 + 10 = 62$ such symbols. So there are $62^5$ such strings.

Problem 2. How many length-10 binary strings exist?

Solution 2. $|\Sigma|^{10} = 2^{10} = 1024$.

Problem 3. How many length-10 binary strings are palindromes?

Solution 3. $|\Sigma|^5 = 2^5 = 32$.

Problem 4. How many length-9 binary strings are palindromes and have a 1 in the middle?

Solution 4. $2^4 = 16$.

Problem 5. How many length-6 strings over $\Sigma = \{a, b, c, \ldots, y, z\}$ have “hello” as a substring?

Solution 5. Such strings either:

- Start with “hello”, followed by another letter.
- Start with a letter, followed by “hello”.

So there are $|\Sigma| + |\Sigma| = 26 + 26 = 52$ such strings.

Problem 6. How many spanning trees does the graph in Figure 1 have?

![Figure 1: An undirected graph.](image)

Solution 6. The graph has $3 \cdot 3 \cdot 4 \cdot 4 = 144$ spanning trees.
2 Binomial Coefficients

Problem 7. How many size-12 subsets does \{1, 2, \ldots, 19, 20\} have?

Solution 7. \binom{20}{12}.

Problem 8. How many size-12 subsets of \{1, 2, \ldots, 19, 20\} contain 3?

Solution 8. There are 11 choices for the remaining items, which come from \{1, 2, \ldots, 19, 20\}—\{3\} (a size-19 set). So there are \binom{19}{11} such subsets.

Problem 9. How many length-7 binary strings have exactly 2 1s?

Solution 9. Each such string has a unique set of 2 locations for the 1s, so \binom{7}{2} such strings.

Problem 10. How many length-5 strings over \Sigma = \{0, 1\} have exactly two 2s?

Solution 10. \binom{5}{2} \cdot 2^5 - 2.

Problem 11. Let \( S \subseteq \mathbb{Z} \) with \(|S| = n\). What is the maximum value of \(|\{a + b : a, b \in S \land a \neq b\}|\)?

Solution 11. This set is the set of all sum of distinct pairs of elements in \( S \). This is at most \( \binom{n}{2} \), since there are \( \binom{n}{2} \) pairs of distinct elements \( a, b \). If \( S = \{10, 100, 1000, \ldots, 10^n\} \), then \(|S| = \binom{n}{2}\), since the sum of every pair of powers of 10 is distinct.

3 Other Binomial Coefficient Stuff

Problem 12. Expand \((x + y)^6\) (use knowledge of binomial coefficients).

Solution 12. \( (x + y)^6 = \binom{6}{0} x^0 y^6 + \binom{6}{1} x^1 y^5 + \binom{6}{2} x^2 y^4 + \binom{6}{3} x^3 y^3 + \binom{6}{4} x^4 y^2 + \binom{6}{5} x^5 y^1 + \binom{6}{6} x^6 y^0 = y^6 + 6xy^5 + 15x^2 y^4 + 20x^3 y^3 + 15x^4 y^2 + 6x^5 y + x^6.\)

Problem 13. Give a closed form (no summation) for \( \sum_{i=0}^{n} 3^i \binom{n}{i} \).

Solution 13. \( \sum_{i=0}^{n} 3^i \binom{n}{i} = \sum_{i=0}^{n} \binom{n}{i} \cdot 3^i \cdot 1^{n-i} = 4^n.\)

4 Counting Permutations with Duplicates

Problem 14. How many permutations of the letters in “comboloco” exist?

Solution 14. \( \frac{9!}{2! 4! 1! 1! 1!}.\)

Problem 15. How many permutations of the letters in “aardvarks” exist?

Solution 15. \( \frac{9!}{3! 2! 1! 1! 1! 1!}.\)

Problem 16. How many permutations of the letters in “bananarama” exist?

Solution 16. \( \frac{9!}{1! 5! 2! 1! 1!}.\)
5 Complement Rule

Problem 17. How many length-$n$ binary strings contain at least 2 1s?

Solution 17. $2^n - n - 1$.

Problem 18. How many length-5 binary strings start or end with a 1?

Solution 18. $2^5 - 2^3$ (2$^3$ is the number of length-5 binary strings that start and end with 0).

Problem 19. How many length-10 binary strings are not palindromes?

Solution 19. $2^{10} - 2^5$.

Problem 20. How many subsets of \{1, 2, \ldots, 9\} contain an even number?

Solution 20. $2^{|\{1,2,\ldots,9\}|} - 2^{|\{1,3,5,7,9\}|} = 2^9 - 2^5$.

6 Balls-In-Bins

Problem 21. How many ways are there to distribute 100 bonus points to 10 students?

Solution 21. \( \binom{100+10-1}{10-1} = \binom{109}{9} \).

Problem 22. How many ways are there to distribute 100 bonus points to 10 students where each student receives at least 1 point?

Solution 22. There are $100 - 10 = 90$ points still to distribute. So \( \binom{90+10-1}{10-1} = \binom{99}{9} \).

Problem 23. How many ways are there to distribute 100 bonus points to 10 students where each student receives at least 2 points?

Solution 23. There are $100 - 2 \cdot 10 = 80$ points still to distribute. So \( \binom{80+10-1}{10-1} = \binom{89}{9} \).

Problem 24. How many ways are there to distribute 100 bonus points to 10 students such that each student receives at least 1 bonus point and exactly one student receives $\leq 3$ bonus points.

Solution 24. Use sum rule to partition point distribution into three types, according to how many bonus points the $\leq 3$ student receives.

- 1 point: remaining students receive 99 points total.
- 2 points: remaining students receive 98 points total.
- 3 points: remaining students receive 97 points total.

In each case, there are two decisions:

- Which student receives $\leq 3$ points (\( \binom{10}{1} = 10 \) options).
- Distribution of remaining (indistinguishable) points across 9 (distinguishable) students.

Thus the total number of ways is $10(\binom{99+9-1}{9-1} + \binom{98+9-1}{9-1} + \binom{97+9-1}{9-1})$. 
7 Pigeonhole Principle

Prove each of the following theorems using the pigeonhole principle.

**Theorem 25.** Every set of 11 integers has two integers with a common digit (e.g., 427 and 902 share the digit 2).

*Proof.* There are only 10 possible digits (0, 1, 2, . . . , 9). Consider the first digit of each of the 11 integers. By pigeonhole principle, two of these digits must be the same. So the integers containing them share a common digit. □

**Theorem 26.** Every length-11 binary string contains two occurrences of the same length-3 substring.

*Proof.* There are $2^3 = 8$ length-3 binary strings. A length-11 string has $11 - 2 = 9$ length-3 substrings. So by pigeonhole principle, two of these substrings must be the same. □

**Theorem 27.** Let $S$ be a set of 3 or more integers. There exist distinct $a, b \in S$ such that $a - b$ is even.

*Proof.* By the pigeonhole principle, any set of at least three integers has two that are even or two that are odd. Call such a pair $a$ and $b$.

If $a$ and $b$ are even, then $a = 2i, b = 2j$ for some $i, j \in \mathbb{Z}$. So $a - b = 2i - 2j = 2(i - j)$. Since $i, j \in \mathbb{Z}, i - j \in \mathbb{Z}$ and $a - b = 2(i - j)$ is an even integer.

If $a$ and $b$ are odd, then $a = 2i + 1, b = 2j + 1$ for some $i, j \in \mathbb{Z}$. So $a - b = (2i + 1) - (2j + 1) = 2(i - j)$. As before, since $i, j \in \mathbb{Z}, a - b = 2(i - j)$ is an even integer. □

**Theorem 28.** Every set of 5 points in an equilateral triangle of side length 1 has two points with distance at most $\frac{1}{2}$.

*Proof.* Partition the triangle into 4 smaller equilateral triangles as seen in Figure 2. By the pigeonhole principle, any set of 5 points in the original equilateral triangle must have two in the same smaller equilateral triangle. Call these two points $p$ and $q$. The smaller triangles have size length $\frac{1}{2}$. So the distance from $p$ to $q$ is at most $\frac{1}{2}$. □

Figure 2: Part of the proof of Theorem 28.