FUNCTIONS

Let $X, Y$ be sets. A subset of $X \times Y$ is a function $f$ from $X$ to $Y$ provided $\forall x \in X (\exists y \in Y ((x, y) \in f) \land \forall y_1, y_2 \in Y ((x, y_1), (x, y_2) \in f) \Rightarrow y_1 = y_2)$.

A function $f$ from $X$ to $Y$ is also written $f : X \rightarrow Y$.

For $x \in X, y \in Y$ with $(x, y) \in f$, $f(x) = y$.

For a function $f : X \rightarrow Y$:
- The domain is $X$.
- The codomain is $Y$.
- The image is $\{f(x) : x \in X\}$.

Example: $f = \{(1, 2), (2, 3), (3, 2), (4, 4)\}$ from $\{1, 2, 3, 4\}$ to $\mathbb{N}$:

\[
\begin{array}{c|c}
1 & 1 \\
2 & 2 \\
3 & 3 \\
4 & 4 \\
\hline
\end{array}
\]

- domain: $\{1, 2, 3, 4\}$
- codomain: $\mathbb{N}$
- image: $\{2, 3, 4\}$
INJECTIVE, SURJECTIVE, & BIJECTIVE FUNCTIONS

A function \( f: X \rightarrow Y \) is \textbf{injective} provided \( \forall x_1, x_2 \in X, f(x_1) = f(x_2) \Rightarrow x_1 = x_2. \)

Examples: 
- \( f: \mathbb{Z} \rightarrow \mathbb{Z} \) with \( f(n) = n+1 \) is \textbf{not injective}.
- \( f: \mathbb{Z} \rightarrow \mathbb{Z} \) with \( f(n) = 2n \) is \textbf{not injective}.
- \( f: \mathbb{Z} \rightarrow \mathbb{N} \cup \{0\} \) with \( f(n) = n^2 \) is \textbf{not injective}.
- \( f: \mathbb{Z} \times \mathbb{N} \rightarrow \mathbb{Q} \) with \( f((p, q)) = \frac{p}{q} \) is \textbf{not injective}.

A function \( f: X \rightarrow Y \) is \textbf{surjective} provided \( \forall y \in Y \exists x \in X, f(x) = y. \)

Examples: 
- \( f: \mathbb{Z} \rightarrow \mathbb{Z} \) with \( f(n) = n+1 \) is \textbf{not surjective}.
- \( f: \mathbb{Z} \rightarrow \mathbb{Z} \) with \( f(n) = 2n \) is \textbf{not surjective}.
- \( f: \mathbb{Q} \rightarrow \mathbb{Q} \) with \( f(x) = 2x \) is \textbf{not surjective}.
- \( f: \mathbb{Z} \rightarrow \mathbb{N} \cup \{0\} \) with \( f(n) = n^2 \) is \textbf{not surjective}.

A function \( f \) is \textbf{bijective} provided \( f \) is injective and surjective.

Examples: 
- \( f: \mathbb{Z} \rightarrow \mathbb{Z} \) with \( f(n) = n+1 \) is \textbf{not bijective}.
- \( f: \mathbb{Z} \rightarrow \mathbb{Z} \) with \( f(n) = -n \) is \textbf{not bijective}.
- \( f: \mathbb{Z} \rightarrow \mathbb{N} \cup \{0\} \) with \( f(n) = n^2 \) is \textbf{not bijective}.
- \( f: \mathbb{Z} \rightarrow \mathbb{Z} \) with \( f(n) = 2n \) is \textbf{not bijective}.
- \( f: \mathbb{Q} \rightarrow \mathbb{Q} \) with \( f(x) = 2x \) is \textbf{bijective}.
- \( f: \mathbb{Z} \times \mathbb{N} \rightarrow \mathbb{Q} \) with \( f((p, q)) = \frac{p}{q} \) is \textbf{not bijective}. 
SOME THEOREMS ABOUT FUNCTIONS

Let $A, B$ be infinite sets. Then $|A| = |B|$ provided there exists a bijective function $f: A \rightarrow B$.

Theorem: $|\{2, 4, 6, 8, \ldots\}| = |\mathbb{N}|$

Proof: By example.
Consider $f: \{2, 4, 6, 8, \ldots\} \rightarrow \mathbb{N}$ with $f(n) = n/2$.
$f$ is bijective.  

Theorem: $|\mathbb{Z}| = |\mathbb{N}|$

Proof: By example.
Consider $f: \mathbb{Z} \rightarrow \mathbb{N}$ with $f(n) = \begin{cases} -2n & : n < 0 \\ 2n+1 & : n \geq 0 \end{cases}$
$f$ is bijective.  


FUNCTION COMPOSITION

Let \( f : X \rightarrow Y \) and \( g : Y \rightarrow Z \) be functions. The composition of \( g \) with \( f \) also written \( g \circ f \), is the function \( g \circ f : X \rightarrow Z \) with \((g \circ f)(x) = g(f(x))\).

Example:

\( f : \mathbb{Z} \rightarrow \mathbb{Z} \) with \( f(n) = 2n \)
\( g : \mathbb{Z} \rightarrow \mathbb{Q} \) with \( g(n) = 3^n \)

\( g \circ f : \mathbb{Z} \rightarrow \mathbb{Q} \) with \((g \circ f)(n) = 3^{2n} = 9^n\)

\( f \circ g \) does not exist
(codomain of \( g \neq \) domain of \( f \))

The inverse of a function \( f : A \rightarrow B \) is the function \( g : B \rightarrow A \) such that \( \forall x \in A ((g \circ f)(x) = x) \), also denoted \( f^{-1} \).

Examples: \( f : \mathbb{Q} \rightarrow \mathbb{Q} \) with \( f(x) = x/2 \), \( f^{-1}(x) = 2x \).

\( g : \mathbb{Z} \rightarrow \mathbb{Z} \) with \( g(n) = 2n \), \( g \) has no inverse.

Theorem: Let \( f \) be a function. Then \( f \) has inverse \( \iff \) \( f \) is bijective.
**COMMON FUNCTIONS IN COMPUTING**

The **floor function** is $f: \mathbb{R} \rightarrow \mathbb{Z}$ with $f(x) = \max\{n \in \mathbb{Z} : n \leq x\}$, also written $\lfloor x \rfloor$.

Examples: $\lfloor 3.14 \rfloor = 3$, $\lfloor 2.718 \rfloor = 2$, $\lfloor -1.5 \rfloor = -2$, $\lfloor 4 \rfloor = 4$

The **ceiling function** is $f: \mathbb{R} \rightarrow \mathbb{Z}$ with $f(x) = \min\{n \in \mathbb{Z} : n \geq x\}$, also written $\lceil x \rceil$.

Examples: $\lceil 3.14 \rceil = 4$, $\lceil 2.718 \rceil = 3$, $\lceil -1.5 \rceil = -1$, $\lceil 4 \rceil = 4$

Let $b \in \mathbb{N}$. The **base-$b$ exponential function** is $f: \mathbb{R} \rightarrow \mathbb{R}^+$ with $f(x) = b^x$.

Let $b \in \mathbb{N}$. The **base-$b$ logarithm function** is the inverse of the base-$b$ exponential function, written $f(x) = \log_b(x)$.

Examples: $\log_2(16) = \log_2(2^4) = 4$

$\log_3(27) = \log_3(3^3) = 3$

$\log_{10}(100) = \log_{10}(10^2) = 2$

A function $f: X \rightarrow Y$ is **increasing** provided

$\forall x_1, x_2 \in X \ (x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2))$

A function $f: X \rightarrow Y$ is **strictly increasing** provided

$\forall x_1, x_2 \in X \ (x_1 < x_2 \Rightarrow f(x_1) < f(x_2))$
EXPONENTS & LOGARITHM IDENTITIES

∀b, c, x, y ∈ ℝ:

\[ b^x \cdot b^y = b^{x+y} \]
\[ (b^x)^y = b^{xy} \]
\[ b^x / b^y = b^{x-y} \]
\[ (bc)^x = b^x \cdot c^x \]

\[ \log_b(x) = \log_b(x) + \log_b(y) \]
\[ \log_b(x^y) = y \cdot \log_b(x) \]
\[ \log_b(x/y) = \log_b(x) - \log_b(y) \]
\[ b^{\log_b(x)} = x \]

\[ \log_c(x) = \log_b(x) / \log_b(c) \] change of base
ALGORITHM EFFICIENCY

A primary concern about code is its efficiency/running time: how long does it take to execute?

Specifically:

- As a function of input size.
- Ignoring platform-specific details:
  CPU clock speed, motherboard bus speeds, CPU instruction set, etc.

Why is this enough? Scaling always wins (eventually):

To compare the scaling of functions, we use asymptotic/big-O notation.
ASYMPTOTIC ANALYSIS

"roughly" means equal, but:

1. Ignoring small values of $n$. asymptotic behavior
2. Ignoring constant factors. platform-independent

Examples: $n^2$ is roughly equal to $\frac{1}{100} n^2$

$n^3 + 7$ is roughly equal to $50n^3 + 500$

10 is roughly equal to $1000000$

$\frac{1}{1000} n$ is not roughly equal to $1000000$

$100n^2$ is not roughly equal to $2^n$

$2^n$ is not roughly equal to $2.00001^n$

We use big-$\Theta$ notation to denote "is roughly equal to":

Examples: $n^2 = \Theta(\frac{1}{1000} n^2)$

$n^3 + 7 = \Theta(50n^3 + 500)$

$10 = \Theta(1000000)$

$\frac{1}{1000} n \neq \Theta(1000000)$

$100n^2 \neq \Theta(2^n)$

$2^n \neq \Theta(3^n)$

Rules of thumb: ignore constants, smaller terms:

Examples:

$3n^3 + n^2 + 1000n + x = \Theta(\frac{1}{100} n^3 + 100n^2 + 200n + 70)$

if remaining stuff is the same, they are equal

$x_n^2 + 3x^5 + 1000n \neq \Theta(x_n^3 + 4x^2 + 90n + 80)$

if remaining stuff is different, they are not equal
Let \( f(n), g(n) \) be functions from \( \mathbb{N} \) to \( \mathbb{R}^+ \). Then \( f(n) = \Theta(g(n)) \) provided \( \exists n_0 \in \mathbb{N}, c_1, c_2 \in \mathbb{R}^+ \) such that
\[
\forall n \geq n_0, \quad c_1 g(n) \leq f(n) \leq c_2 g(n).
\]

Theorem: \( 100n + 7 = \Theta(n) \)

Proof: Let \( n_0 = 10, c_1 = 1, c_2 = 200 \).
\[
\forall n \geq 1, \quad n \leq 100n + 7.
\]
\[
\forall n \geq 10, \quad 100n + 7 \leq 100n + 100n = 200n.
\]
So \( \forall n \geq n_0 = 10, \quad c_1 n \leq 100n + 7 \).
\[
\forall n \geq 10, \quad 100n + 7 \leq c_2 n.
\]
So \( \forall n \geq n_0, \quad c_1 n \leq 100n + 7 \leq c_2 n. \)

Thus \( 100n + 7 = \Theta(n) \). \( \Box \)
Two more notations like big-$\Theta$:

Big-$\Theta$: $f(n) = \Theta(g(n))$  “$f(n)$ is roughly equal to $g(n)$”  $f(n) = g(n)$

Big-$O$: $f(n) = O(g(n))$  “$f(n)$ is roughly at most $g(n)$”  $f(n) \leq g(n)$

Big-$\Omega$: $f(n) = \Omega(g(n))$  “$f(n)$ is roughly at least $g(n)$”  $f(n) \geq g(n)$

They have similar formal definitions:

Let $f(n), g(n)$ be functions from $\mathbb{N}$ to $\mathbb{R}^+$. Then $f(n) = O(g(n))$ provided $\exists n_0 \in \mathbb{N}, c \in \mathbb{R}^+$ such that $\forall n \geq n_0, f(n) \leq c \cdot g(n)$.

Let $f(n), g(n)$ be functions from $\mathbb{N}$ to $\mathbb{R}^+$. Then $f(n) = \Omega(g(n))$ provided $\exists n_0 \in \mathbb{N}, c \in \mathbb{R}^+$ such that $\forall n \geq n_0, c \cdot g(n) \leq f(n)$.
Theorem: Let $f(n), g(n)$ be functions from $\mathbb{N}$ to $\mathbb{R}^+$. Then $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n)) \land f(n) = \Omega(g(n))$. 

Theorem: Let $f(n), g(n)$ be functions from $\mathbb{N}$ to $\mathbb{R}^+$. Then $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$. 

\[ g(n) \quad f(n) \]