FUNCTIONS

Let $X, Y$ be sets. A subset of $X \times Y$ is a **function** $f$ from $X$ to $Y$ provided $\forall x \in X (\exists y \in Y ((x, y) \in f) \land \forall y_1, y_2 \in Y ((x, y_1), (x, y_2) \in f) \Rightarrow y_1 = y_2)$.

A function $f$ from $X$ to $Y$ is also written $f : X \rightarrow Y$. For $x \in X, y \in Y$ with $(x, y) \in f$, $f(x) = y$.

For a function $f : X \rightarrow Y$:

- The **domain** is $X$.
- The **codomain** is $Y$.
- The **image** is $\{f(x) : x \in X\}$.

Example: $f = \{(1, 2), (2, 3), (3, 2), (4, 4)\}$ from $\{1, 2, 3, 4\}$ to $\mathbb{N}$:

- **Domain**: $\{1, 2, 3, 4\}$
- **Codomain**: $\mathbb{N}$
- **Image**: $\{2, 3, 4\}$
INJECTIVE, SURJECTIVE, & BIJECTIVE FUNCTIONS

A function \( f: X \to Y \) is **injective** provided \( \forall x_1, x_2 \in X, f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \).

**Examples:**
- \( f: \mathbb{Z} \to \mathbb{Z} \) with \( f(n) = n + 1 \)
- \( f: \mathbb{Z} \to \mathbb{Z} \) with \( f(n) = 2n \)
- \( f: \mathbb{Z} \to \mathbb{N} \cup \{0\} \) with \( f(n) = n^2 \) is **not** injective.
- \( f: \mathbb{Z} \times \mathbb{N} \to \mathbb{Q} \) with \( f((p, q)) = \frac{p}{q} \) is **not** injective.

A function \( f: X \to Y \) is **surjective** provided \( \forall y \in Y \exists x \in X, f(x) = y \).

**Examples:**
- \( f: \mathbb{Z} \to \mathbb{Z} \) with \( f(n) = n + 1 \)
- \( f: \mathbb{Z} \to \mathbb{Z} \) with \( f(n) = 2n \) is **not** surjective.
- \( f: \mathbb{Q} \to \mathbb{Q} \) with \( f(x) = 2x \)
- \( f: \mathbb{Z} \to \mathbb{N} \cup \{0\} \) with \( f(n) = n^2 \) is **not** surjective.

A function \( f \) is **bijective** provided \( f \) is injective and surjective.

**Examples:**
- \( f: \mathbb{Z} \to \mathbb{Z} \) with \( f(n) = n + 1 \)
- \( f: \mathbb{Z} \to \mathbb{Z} \) with \( f(n) = -n \)
- \( f: \mathbb{Z} \to \mathbb{N} \cup \{0\} \) with \( f(n) = n^2 \) is **not** bijective.
- \( f: \mathbb{Z} \to \mathbb{Z} \) with \( f(n) = 2n \) is **not** bijective.
- \( f: \mathbb{Q} \to \mathbb{Q} \) with \( f(x) = 2x \) is bijective.
- \( f: \mathbb{Z} \times \mathbb{N} \to \mathbb{Q} \) with \( f((p, q)) = \frac{p}{q} \) is **not** bijective.
SOME THEOREMS ABOUT FUNCTIONS

Let $A, B$ be infinite sets. Then $|A| = |B|$ provided there exists a bijective function $f: A \to B$.

Theorem: $|\{2, 4, 6, 8, \ldots\}| = \aleph_0$

Proof: By example.
Consider $f: \{2, 4, 6, 8, \ldots\} \to \mathbb{N}$ with $f(n) = n/2$.
$f$ is bijective.

$|\mathbb{R}| \neq |\mathbb{Q}| = |\mathbb{N}| = |\mathbb{Z}|$

Theorem: $|\mathbb{Z}| = |\mathbb{N}|$

Proof: By example.
Consider $f: \mathbb{Z} \to \mathbb{N}$ with $f(n) = \begin{cases} -2n & : n < 0 \\ 2n+1 & : n \geq 0 \end{cases}$
$f$ is bijective.
**FUNCTION COMPOSITION**

Let \( f: X \rightarrow Y \) and \( g: Y \rightarrow Z \) be functions. The **composition** of \( g \) with \( f \) also written \( g \circ f \) is the function \( g \circ f: X \rightarrow Z \) with \((g \circ f)(x) = g(f(x))\).

**Example:**

\( f: \mathbb{Z} \rightarrow \mathbb{Z} \) with \( f(n) = 2n \)

\( g: \mathbb{Z} \rightarrow \mathbb{Q} \) with \( g(n) = 3^n \)

\( g \circ f: \mathbb{Z} \rightarrow \mathbb{Q} \) with \((g \circ f)(n) = 3^{2n} = 9^n\)

\( f \circ g \) does not exist

(codomain of \( g \neq \) domain of \( f \))

The **inverse** of a function \( f: A \rightarrow B \) is the function \( g: B \rightarrow A \) such that \( \forall x \in A, ((g \circ f)(x) = x) \), also denoted \( f^{-1} \).

**Examples:**

\( f: \mathbb{Q} \rightarrow \mathbb{Q} \) with \( f(x) = x/2 \), \( f^{-1}(x) = 2x \).

\( g: \mathbb{Z} \rightarrow \mathbb{Z} \) with \( g(n) = 2n \), \( g \) has no **inverse**.

**Theorem:** Let \( f \) be a function. Then \( f \) has inverse \( \iff \) \( f \) is bijective.
COMMON FUNCTIONS IN COMPUTING

The floor function is \( f : \mathbb{R} \to \mathbb{Z} \) with \( f(x) = \max(\{n \in \mathbb{Z} : n \leq x\}) \), also written \( \lfloor x \rfloor \).

Examples: \( \lfloor 3.14 \rfloor = 3 \), \( \lfloor 2.718 \rfloor = 2 \), \( \lfloor -1.5 \rfloor = -2 \), \( \lfloor 4 \rfloor = 4 \)

The ceiling function is \( f : \mathbb{R} \to \mathbb{Z} \) with \( f(x) = \min(\{n \in \mathbb{Z} : n \geq x\}) \), also written \( \lceil x \rceil \).

Examples: \( \lceil 3.14 \rceil = 4 \), \( \lceil 2.718 \rceil = 3 \), \( \lceil -1.5 \rceil = -1 \), \( \lceil 4 \rceil = 4 \)

Let \( b \in \mathbb{N} \). The base-\( b \) exponential function is \( f : \mathbb{R} \to \mathbb{R}^+ \) with \( f(x) = b^x \).

Let \( b\in\mathbb{N} \). The base-\( b \) logarithm function is the inverse of the base-\( b \) exponential function, written \( f(x) = \log_b(x) \).

Examples:
\[
\log_2(16) = \log_2(2^4) = 4 \\
\log_3(27) = \log_3(3^3) = 3 \\
\log_{10}(100) = \log_{10}(10^2) = 2
\]

A function \( f : X \to Y \) is increasing provided
\[
\forall x_1, x_2 \in X (x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2))
\]

A function \( f : X \to Y \) is strictly increasing provided
\[
\forall x_1, x_2 \in X (x_1 < x_2 \Rightarrow f(x_1) < f(x_2))
\]
## EXPONENTS & LOGARITHM IDENTITIES

\[ \forall b, c, x, y \in \mathbb{R}: \]

\begin{align*}
\text{b}^x \cdot \text{b}^y &= \text{b}^{x+y} & \log_b (xy) &= \log_b (x) + \log_b (y) \\
(b^x)^y &= b^{xy} & \log_b (x^y) &= y \cdot \log_b (x) \\
\frac{b^x}{b^y} &= b^{x-y} & \log_b (x/y) &= \log_b (x) - \log_b (y) \\
(b^c)^x &= b^{x \cdot c} & b^{\log_b (x)} &= x \\
\end{align*}

\[ \log_c (x) = \frac{\log_b (x)}{\log_b (c)} \quad \text{change of base} \]
ALGORITHM EFFICIENCY

A primary concern about code is its efficiency/running time: how long does it take to execute?

Specifically:

- As a function of input size.
- Ignoring platform-specific details: CPU clock speed, motherboard bus speeds, CPU instruction set, etc.

Why is this enough? Scaling always wins (eventually):

Comparing the scaling of functions is asymptotic analysis.
ASYMPTOTIC ANALYSIS

Compare functions "roughly", meaning:

1. Ignoring small values of \( n \). \( n^2 \geq 5n \) (ignoring \( n \leq 4 \))
2. Ignoring constant factors. \( n^2 = 8n^2 \) (ignoring)

Use special big-\(O/\Omega/\Theta\) notation:

- \( f(n) = O(g(n)) \) provided roughly \( f(n) \leq g(n) \).
- \( f(n) = \Omega(g(n)) \) provided roughly \( f(n) \geq g(n) \).
- \( f(n) = \Theta(g(n)) \) provided roughly \( f(n) = g(n) \).

Examples:
\[ n^2 = O(n^3) \]
\[ n^2 = O(n^{100}) \]
\[ \log(n) = O(\log(n)) \]
\[ n^2 = \Omega(50n^2) \]
\[ n^3 = \Omega(n^3 + 100) \]
\[ 2^n = \Omega(n^{100}) \]
\[ 2^n = \Omega(1.9^n) \]
\[ n = \Theta(7n + 500) \]
\[ 8 = \Theta(8000) \]
\[ n^4 = \Theta(n^4 + 300n^3) \]
\[ \lfloor x \rfloor = \Theta(\lfloor x \rfloor) \]

Rules of thumb:
- Ignore all but largest term.
- \( \forall c \in \mathbb{R}^+ : c^n \gg n^c \gg (\log(n))^c \gg c \).
BIG-O FORMALITIES

Let $f(n), g(n)$ be functions from $\mathbb{N}$ to $\mathbb{R}$. Then $f(n) = O(g(n))$ provided $\exists c \in \mathbb{R}^+, n_0 \in \mathbb{N}$ such that $\forall n \geq n_0, f(n) \leq c \cdot g(n)$.

Theorem: $n = O(0.1n^2)$

Proof: By example.

Let $c = 5, n_0 = 4$.

Then $\forall n \geq n_0 = 4$:

$n \leq 4$  

$n \cdot n \geq 4n$ (since $n > 0$)  

$0.25n^2 \geq n$  

$25 \cdot 0.1n^2 \geq n$  

$5 \cdot 0.1n^2 \geq n$  

$c \cdot 0.1n^2 \geq n$

So $\exists c \in \mathbb{R}^+, n_0 \in \mathbb{N}$ such that $\forall n \geq n_0, n \leq c \cdot 0.1n^2$. \qed
BIG-Ω FORMALITIES

Let \( f(n), g(n) \) be functions from \( \mathbb{N} \) to \( \mathbb{R} \). Then \( f(n) = \Omega(g(n)) \) provided there exist \( c \in \mathbb{R}^+ \), \( n_0 \in \mathbb{N} \) such that \( \forall n \geq n_0 \), \( f(n) \geq c \cdot g(n) \).

\[
\begin{align*}
&\text{Theorem: } n^3 = \Omega(n^2 + 100) \\
&\text{Proof: } \text{By example.} \\
&\quad \text{Let } c=1, n_0 = 10: \\
&\quad \text{Then } \forall n \geq n_0 = 10: \\
&\quad \quad n \geq 10 \\
&\quad \quad n^3 = 10n^2 (\text{since } n^2 > 0) \\
&\quad \quad \geq n^2 + 9n^2 \geq n^2 + 9 \cdot 10^2 (\text{since } n \geq 10) \\
&\quad \quad \geq n^2 + 900 \geq n^2 + 100 \\
&\quad \quad \geq 1(n^3 + 100) = c(n^2 + 100) \\
&\quad \therefore \exists c \in \mathbb{R}^+, n_0 \in \mathbb{N} \text{ such that } \forall n \geq n_0, \ n^3 \geq c(n^2 + 100). \quad \blacksquare
\end{align*}
\]
BIG-Θ FORMALITIES

Let $f(n), g(n)$ be functions from $\mathbb{N}$ to $\mathbb{R}$. Then $f(n) = \Theta(g(n))$ provided $\exists c, c_2 \in \mathbb{R}^+, n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, $c_1 g(n) \leq f(n) \leq c_2 g(n)$.

**Theorem:** $n = \Theta(100n + 7)$

**Proof:** By example.

Let $c_1 = \frac{1}{100}$, $c_2 = 1$, $n_0 = 1$.

Then $\forall n \geq n_0 = 1$:

1. $1 \leq 100$
2. $n \leq 100n$ (since $n > 0$)
3. $n \leq 100n + 7$
4. $n \leq 100n + 7$
5. $n \leq c_2 (100n + 7)$

Also, $\forall n \geq n_0 = 1$:

1. $n \leq n$
2. $\frac{1}{100} \cdot 100n \leq n$
3. $\frac{1}{100} (100n + 7n) \leq n$
4. $\frac{1}{100} (100n + 7) \leq n$
5. $c_1 (100n + 7) \leq n$

So $\exists c_1, c_2 \in \mathbb{R}^+, n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, $c_1 (100n + 7) \leq n \leq c_2 (100n + 7)$. □