1 Graph Representations

Problem 1. Drawing the following graph: \( G = (\{a, b, c, d\}, \{\{a, b\}, \{c, d\}, \{a, d\}, \{a, c\}\}) \). Is \( G \) directed or undirected? How many vertices and edges does \( G \) have?

Problem 2. Consider the graph \( G = (V, E) \) in Figure 1. Give \( V \) and \( E \).

Problem 3. Consider the graph \( G = (V, E) \) in Figure 2. Give \( V \) and \( E \).

2 Neighbors and Degrees

Problem 4. Consider a undirected graph \( G = (V, E) \) with the following properties:

- \(|V| = 5\) and \( V = \{a, b, c, d, e\} \).
- Vertices \( a \) and \( e \) are not neighbors.
- Vertex \( a \) has degree 3.
- Vertices \( b, c, d \) have degree 2.
- The total degree of \( G \) is 12.

Give \( \deg(e) \), \(|E|\), and \( \Delta(G) \). Draw \( G \).

Problem 5. Consider a directed graph \( G = (V, E) \) such that:
• $V = \{a, b, c, d, e\}$.
• $|E| = 10$.
• $\forall v \in V, \deg^+(v) = \deg^-(v)$.
• $(a, c), (a, e), (d, c) \in E$.

Give the total degree of $G$ and $\Delta(G)$. Draw $G$.

**Problem 6.** Consider a directed graph $G = (V, E)$ with the following properties:

- $|V| = 6$ and $V = \{a_1, a_2, b_1, b_2, c_1, c_2\}$.
- Vertices $a_1, a_2$ have in-degree 1 and out-degree 0.
- Vertices $b_1, b_2$ have in-degree 2 and out-degree 1.
- Vertices $c_1, c_2$ have out-degree 2.

Give $\deg^-(c_1), \deg^-(c_2), \Delta(G)$, and $|E|$. Draw $G$.

### 3 Walks, Circuits, Paths, Cycles, and Hamiltonicity

**Problem 7.** Consider the graph $G = (V, E)$ in Figure 3. Give the following:

- A walk of length 8 in $G$.
- A circuit of length 8 in $G$.
- A path of length 8 in $G$.
- A cycle of length 3 in $G$.
- A cycle of length 4 in $G$.

Is $G$ Hamiltonian?

![Figure 3: The graph considered in Problem 7.](image)
Problem 8. Consider the graph $G = (V, E)$ in Figure 4. Give the following:

- A walk of length 8 in $G$.
- A circuit of length 8 in $G$.
- A path of length 5 in $G$.
- A cycle of length 4 in $G$.
- A cycle of length 6 in $G$.

Is $G$ Hamiltonian?

![Figure 4: The graph considered in Problem 8.](image-url)

4 Connectivity

Problem 9. Consider an undirected graph $G = (V, E)$ such that:

- $|V| = 7$.
- $\Delta(G) = 2$.
- The total degree of $G$ is 14.
- $G$ is not connected.

Give $|E|$. Draw $G$.

Problem 10. Consider the graphs $G_1$ and $G_2$ in Figures 3 and 4. Are $G_1$, $G_2$ strongly connected?

5 $k$-Connectivity and $k$-Edge-Connectivity

Problem 11. Consider the graph $G$ in Figure 2. List all $k$ for which $G$ is $k$-connected. Do the same for $k$-edge-connected.

Problem 12. Consider the graph $G$ in Figure 5. List all $k$ for which $G$ is $k$-connected. Do the same for $k$-edge-connected.

Problem 13. Consider the graph $G$ in Figure 6. List all $k$ for which $G$ is $k$-connected. Do the same for $k$-edge-connected.
6 Planarity and Coloring

Problem 14. Is the graph $G = (V, E)$ in Figure 7 planar? If so, list every edge $e$ such that $G = (V, E \cup \{e\})$ is not planar. If not, list every edge $e$ such that $G = (V, E - \{e\})$ is planar.

Problem 15. Find a 4-coloring of the graph in Figure 7.

Problem 16. Draw two graphs:
   1. A planar graph that is not 3-colorable.
   2. A non-planar 3-colorable graph.

Problem 17. Draw a planar 3-edge-connected undirected graph with 8 vertices.

Problem 18. Let $G$ be a graph. Prove that if $\Delta(G) \leq 3$, then $G$ is 4-colorable.
7 Distance

Problem 19. Consider the graph \( G \) in Figure 8. Give the following: \( d(v_1, v_2), d(v_1, v_3), d(v_3, v_4), d(v_5, v_1) \). Give the diameter of \( G \).

![Figure 8: The graph considered in Problem 19.](image)

Problem 20. Consider the graph \( G \) in Figure 9. Give the following: \( d(v_1, v_2), d(v_1, v_3), d(v_2, v_4), d(v_4, v_2), d(v_5, v_6) \). Give the diameter of \( G \).

![Figure 9: The graph considered in Problem 20.](image)

Problem 21. Consider the graph \( G \) in Figure 10. Give the following: \( d(v_1, v_2), d(v_1, v_3), d(v_2, v_4), d(v_5, v_3), d(v_2, v_5) \). Give the diameter of \( G \).

![Figure 10: The graph considered in Problem 21.](image)

Problem 22. Consider the graph \( G \) in Figure 11. Give the following: \( d(v_5, v_1), d(v_1, v_5), d(v_8, v_1), d(v_7, v_4), d(v_9, v_8), d(v_6, v_1), d(v_5, v_3) \). Give the diameter of \( G \).

Problem 23. Give an undirected weighted graph \( G = (V, E) \) such that \( \forall v_i, v_j \in V, d(v_i, v_j) \) is even, \( |V| = 4 \) and \( G \) has diameter 6.
Figure 11: The graph considered in Problem 22.